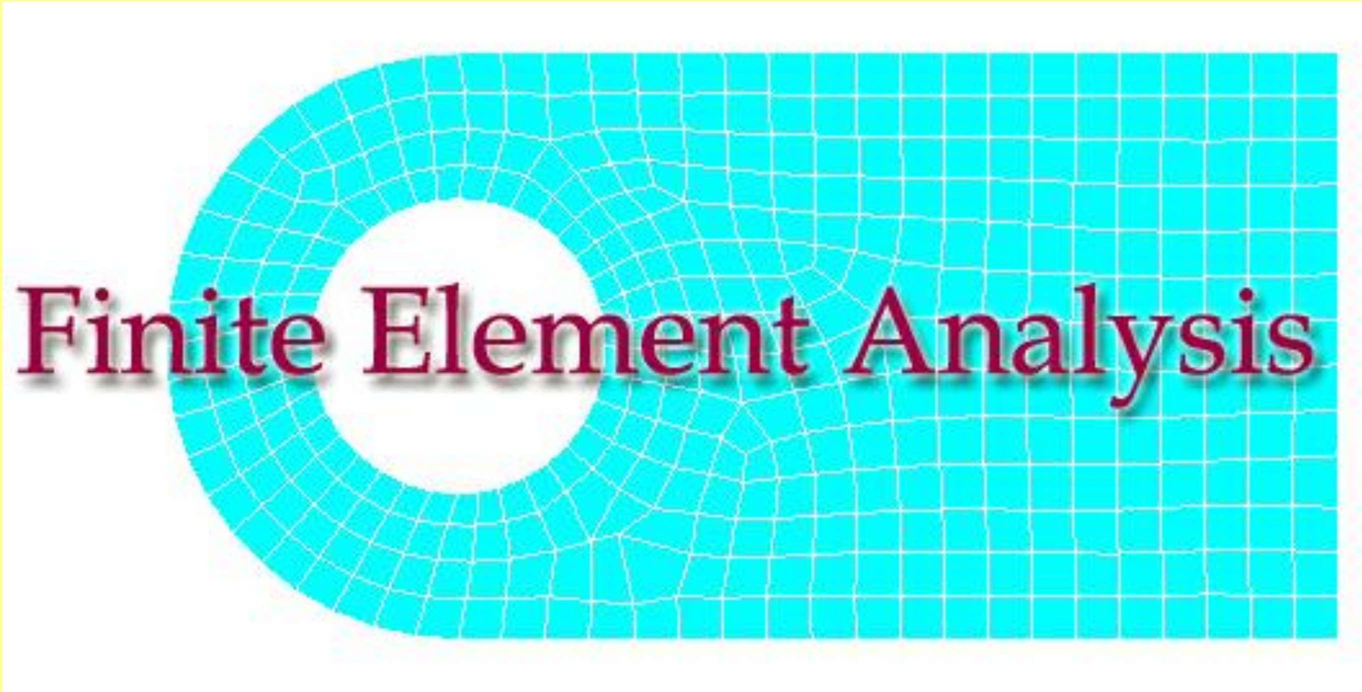

Finite Element Analysis

Why FEM ?

Predictive Method of Analysis

Vs

Experimental Analysis

What is FEM ?

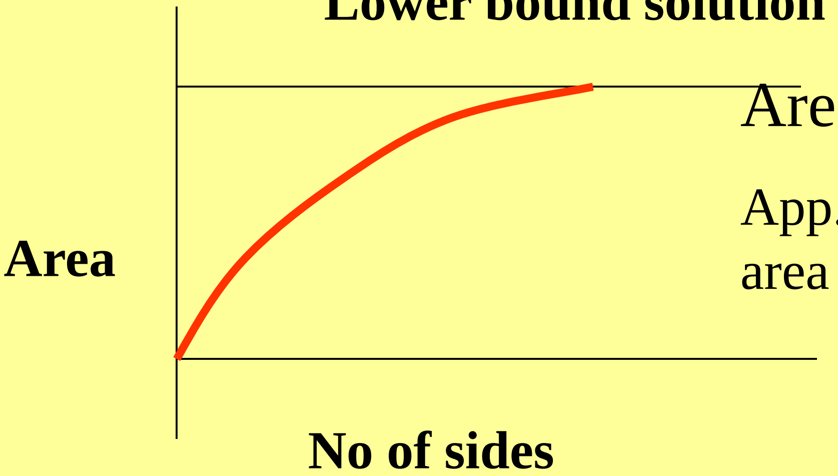
- **Determination of the solution for a complicated problem by replacing it by a simpler one.**

- **Geometrically complex domain represented as a collection of smaller manageable domains.**

- **Solution to these geometrically simple domains is easier.**

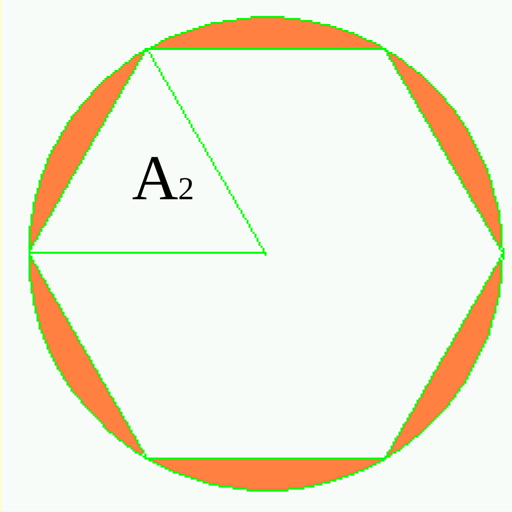
- **Replacing the original complex geometry as an assemblage of smaller simple geometry will result in only an approximate solution.**

Lower bound solution

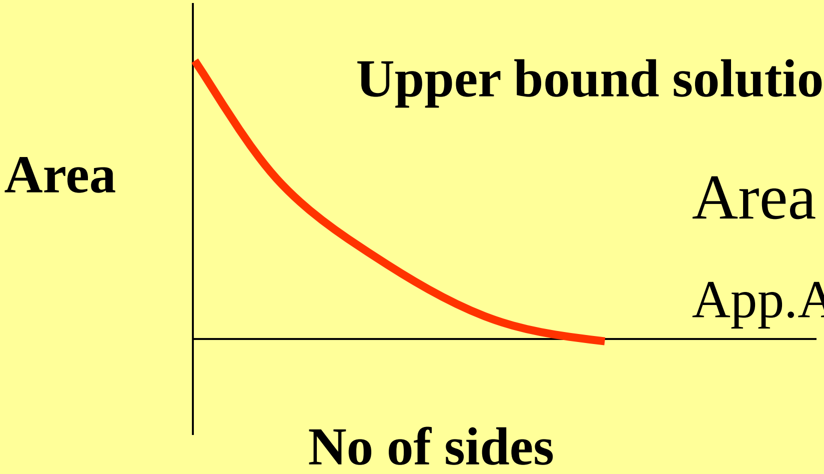


Area = 6 x A_2

App.Area < actual area

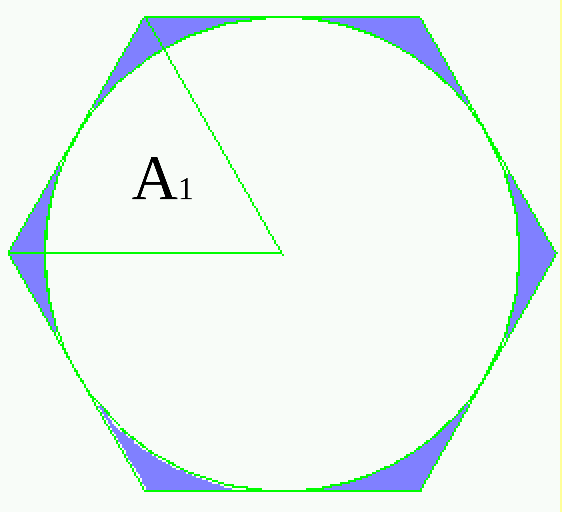


Upper bound solution

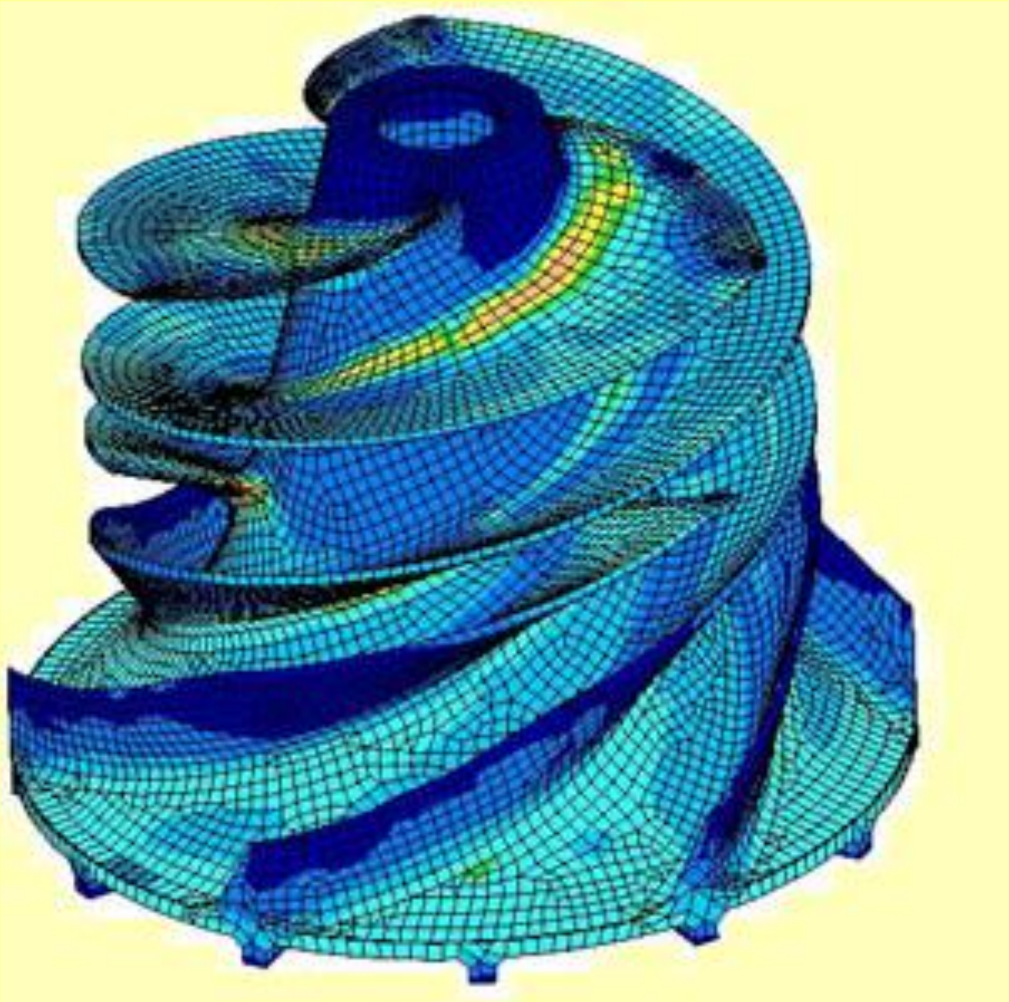


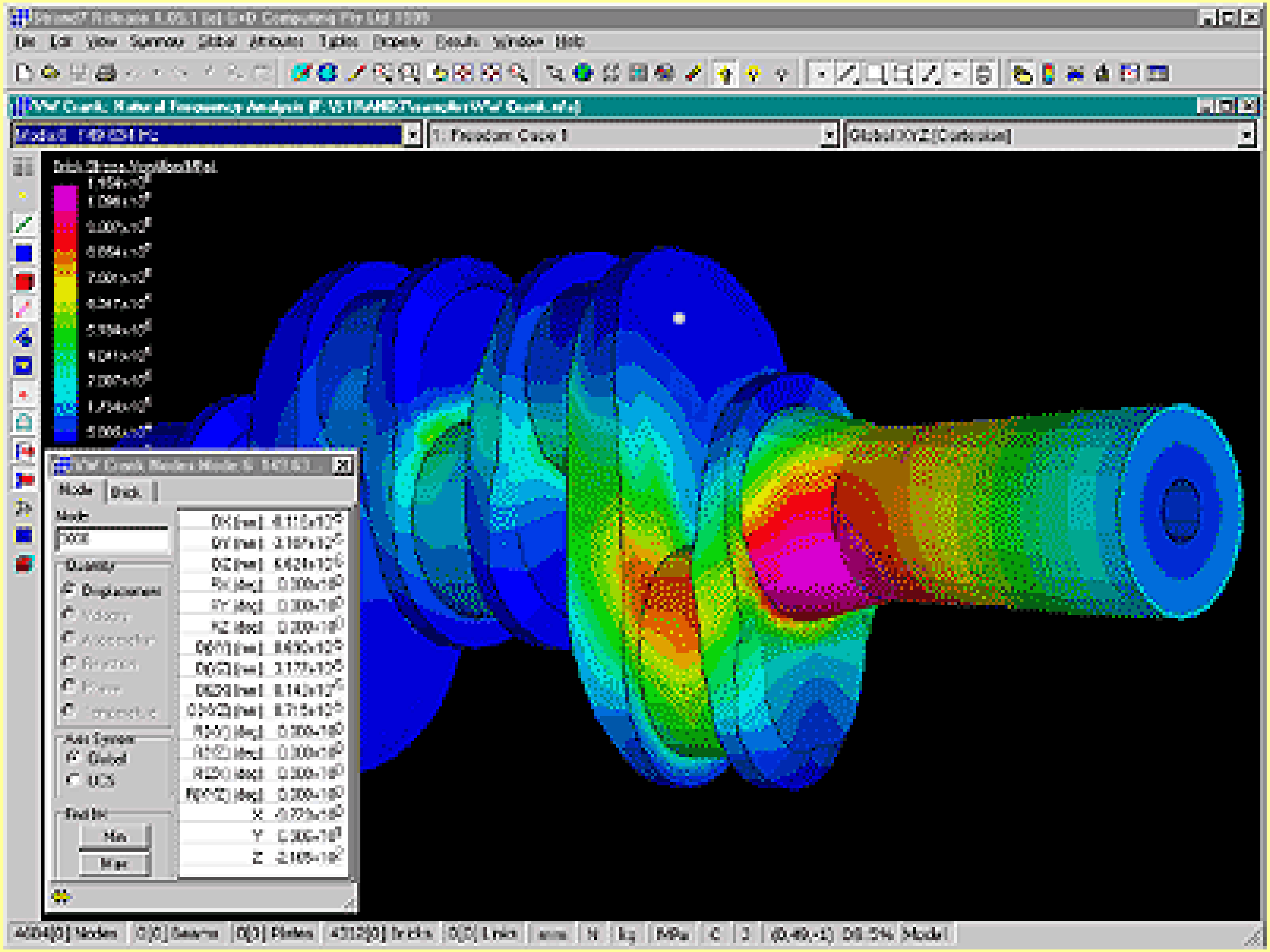
Area = 6 x A_1

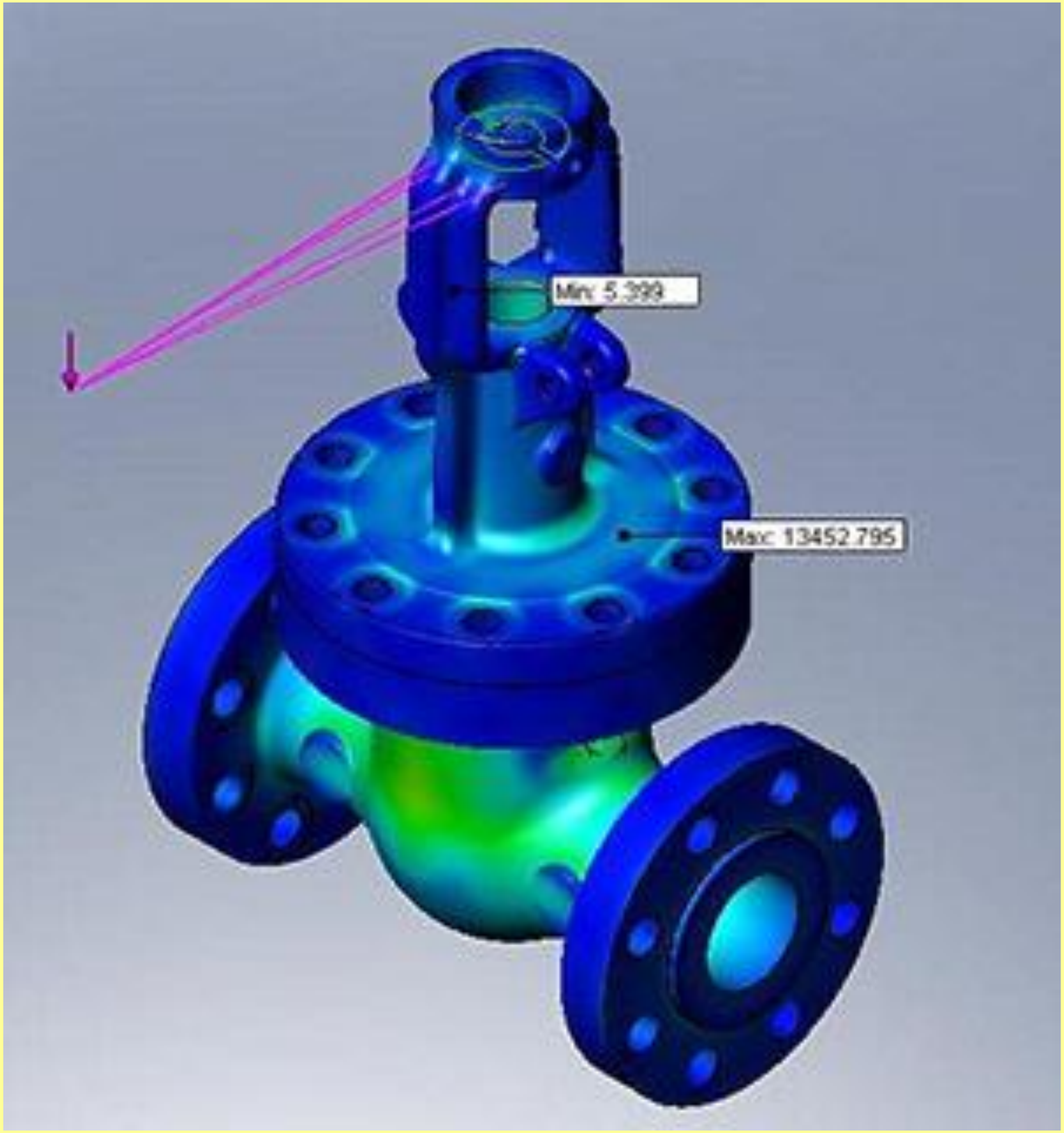
App.Area > actual area

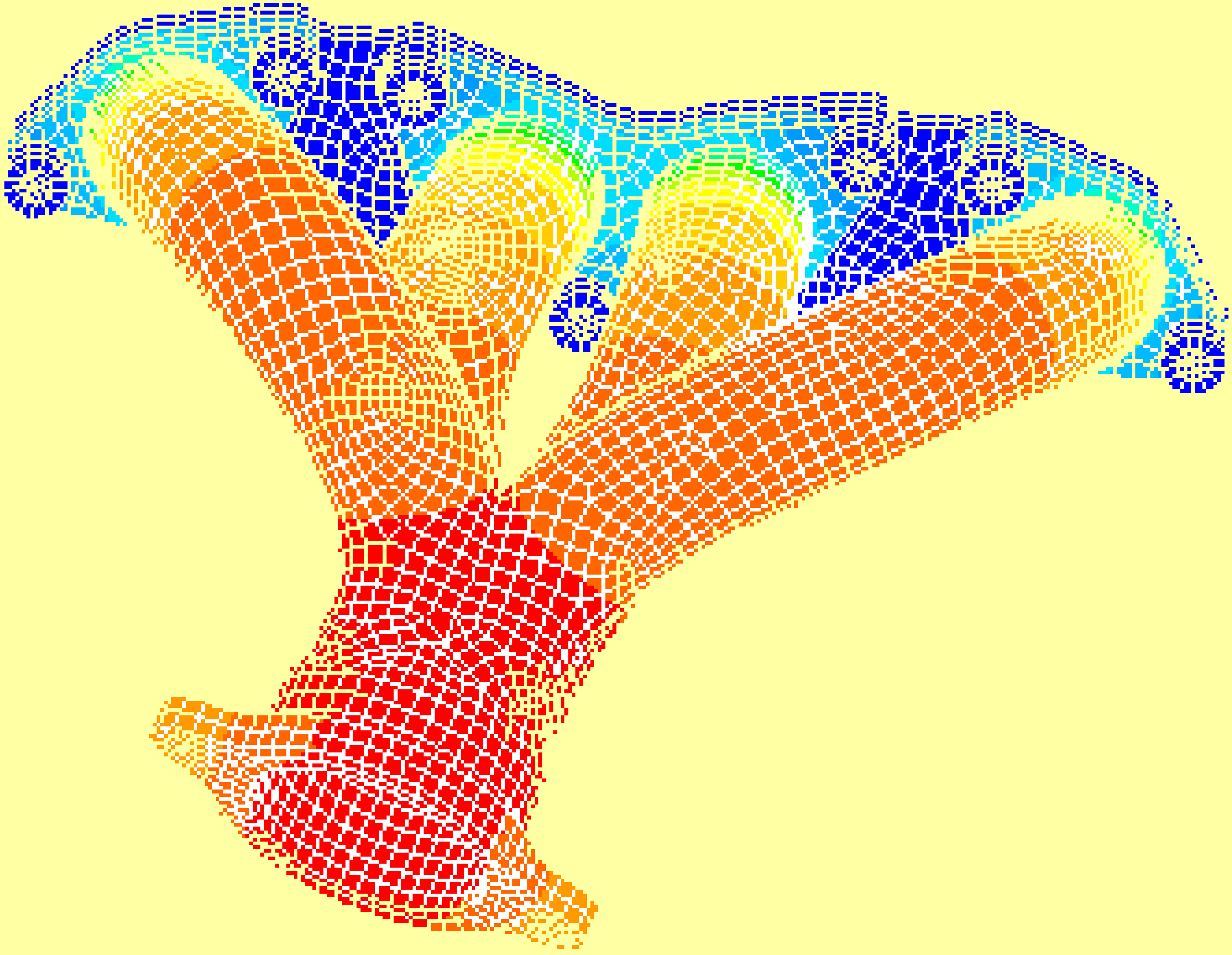


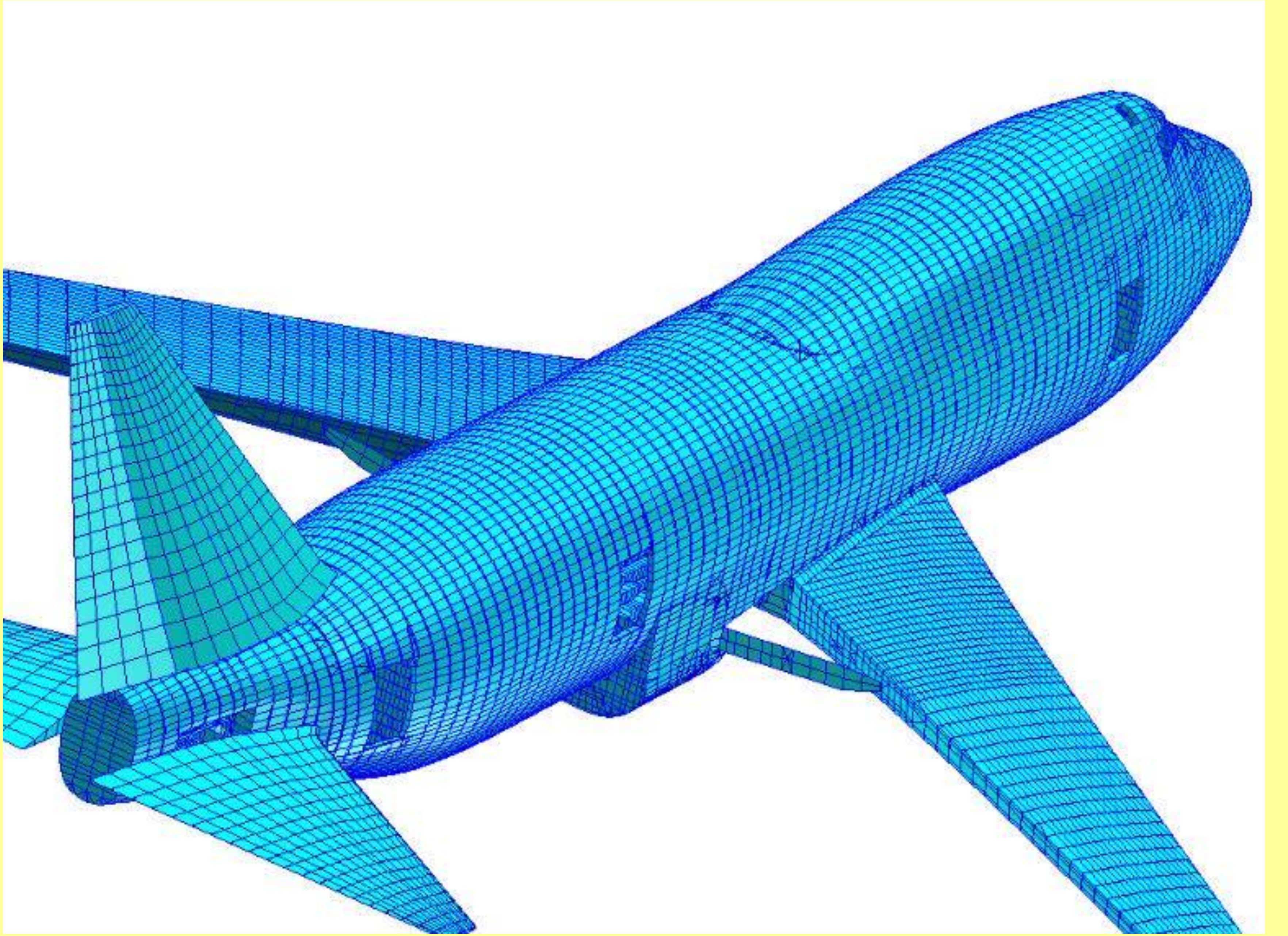
Where FEM ?

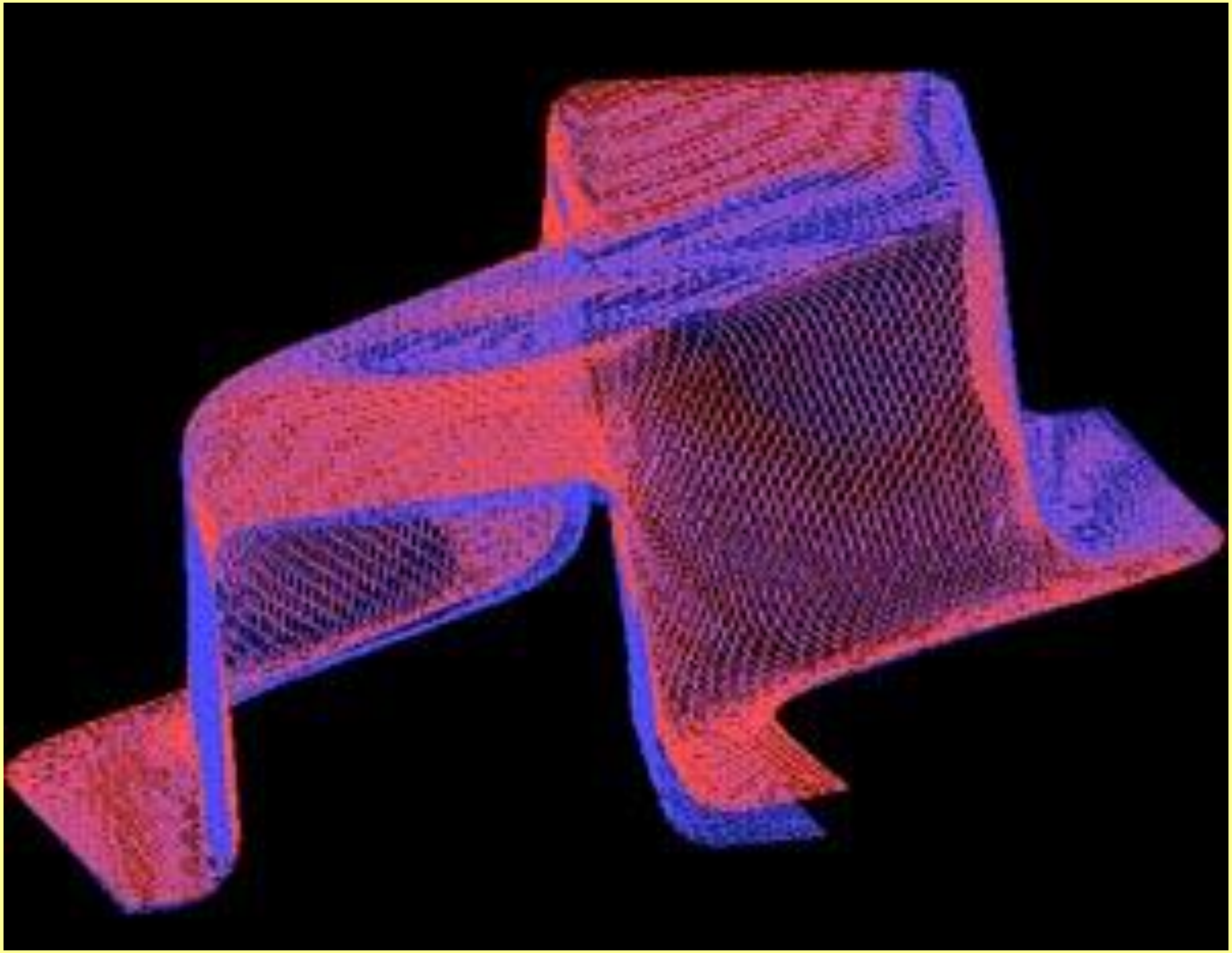


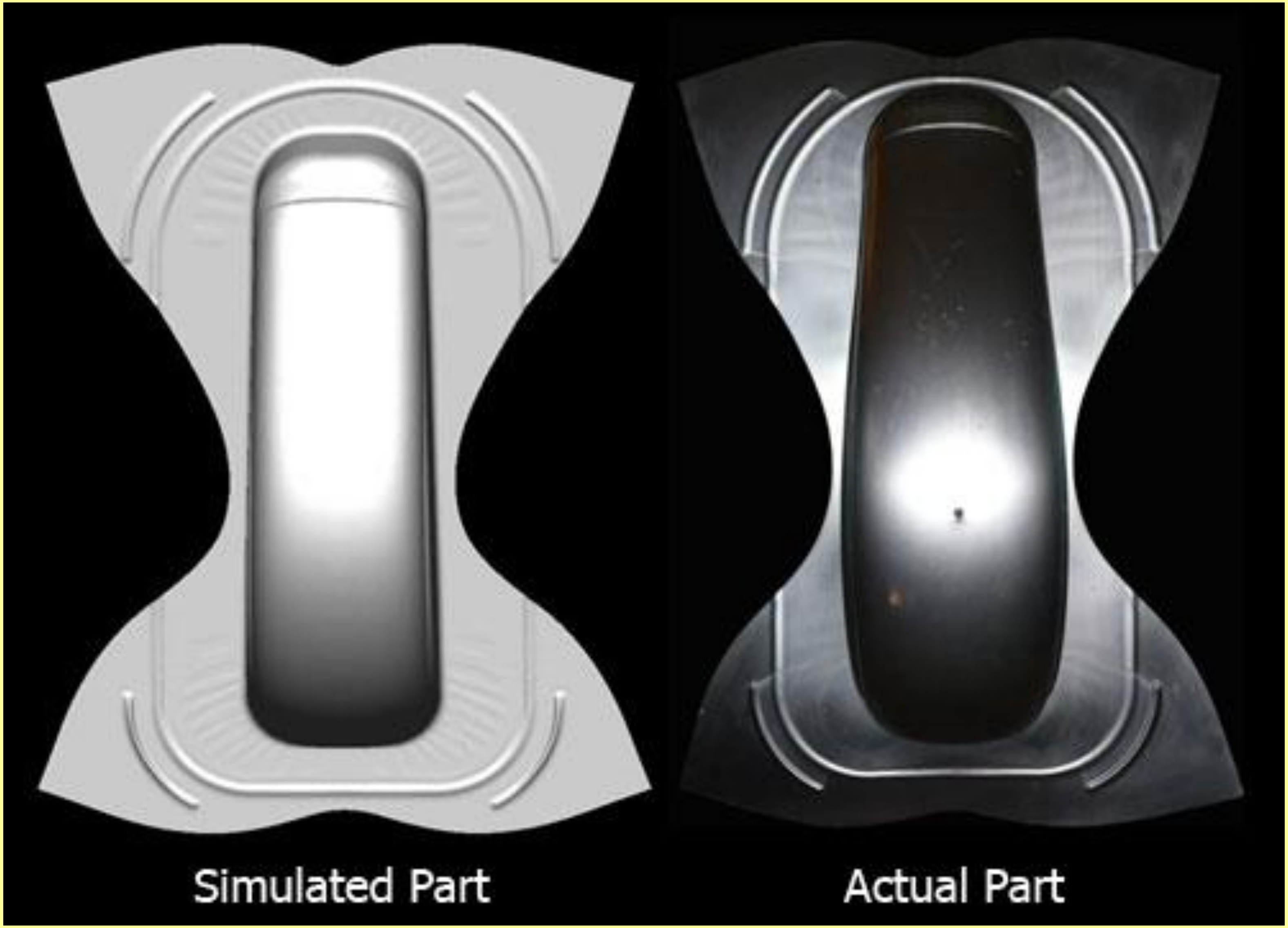

1: Freedom Case 1
Mode
Brick
Quantity
Displacement
Global
UCS
Min
Max

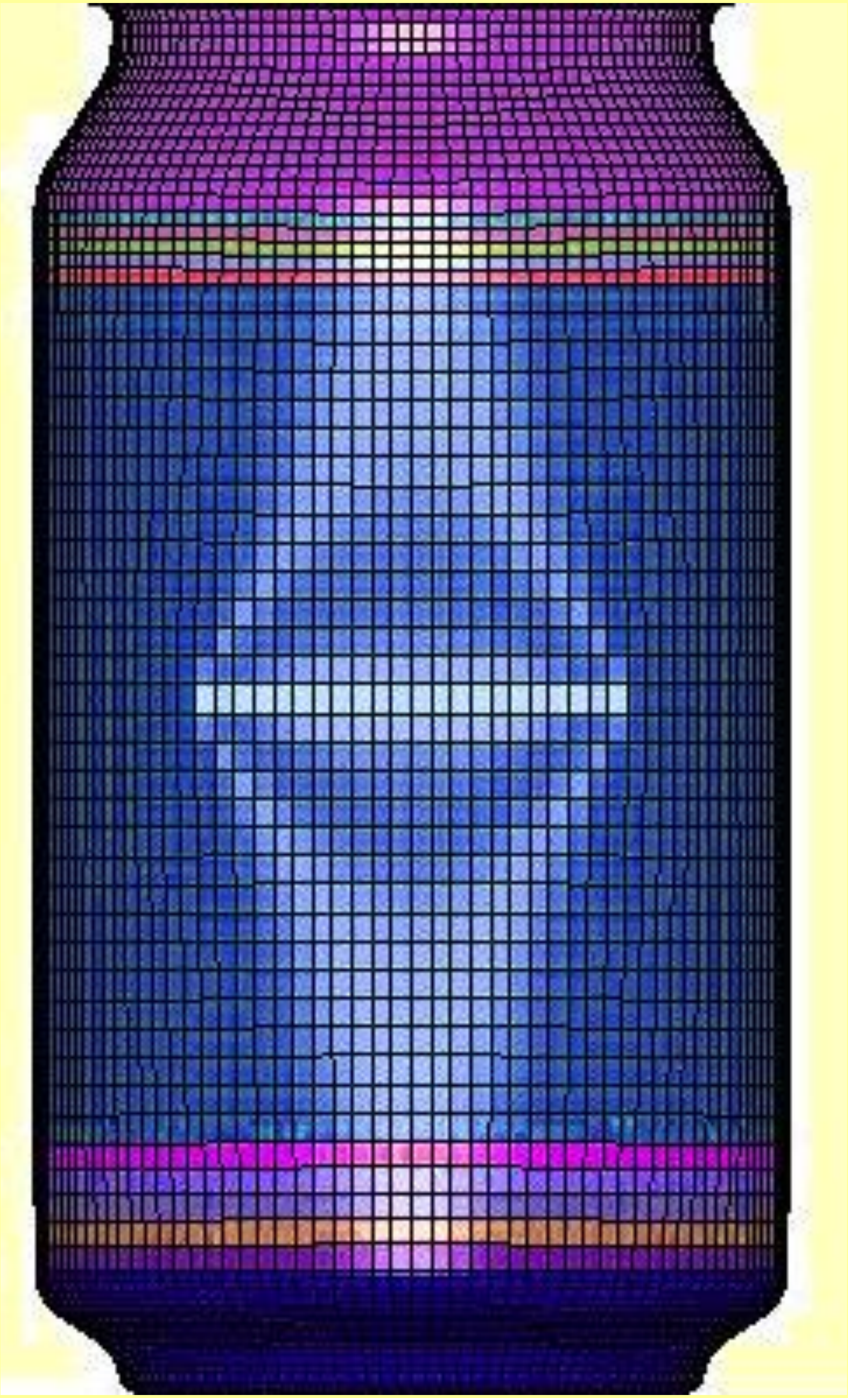

Min: 5.399
Max: 13452.795

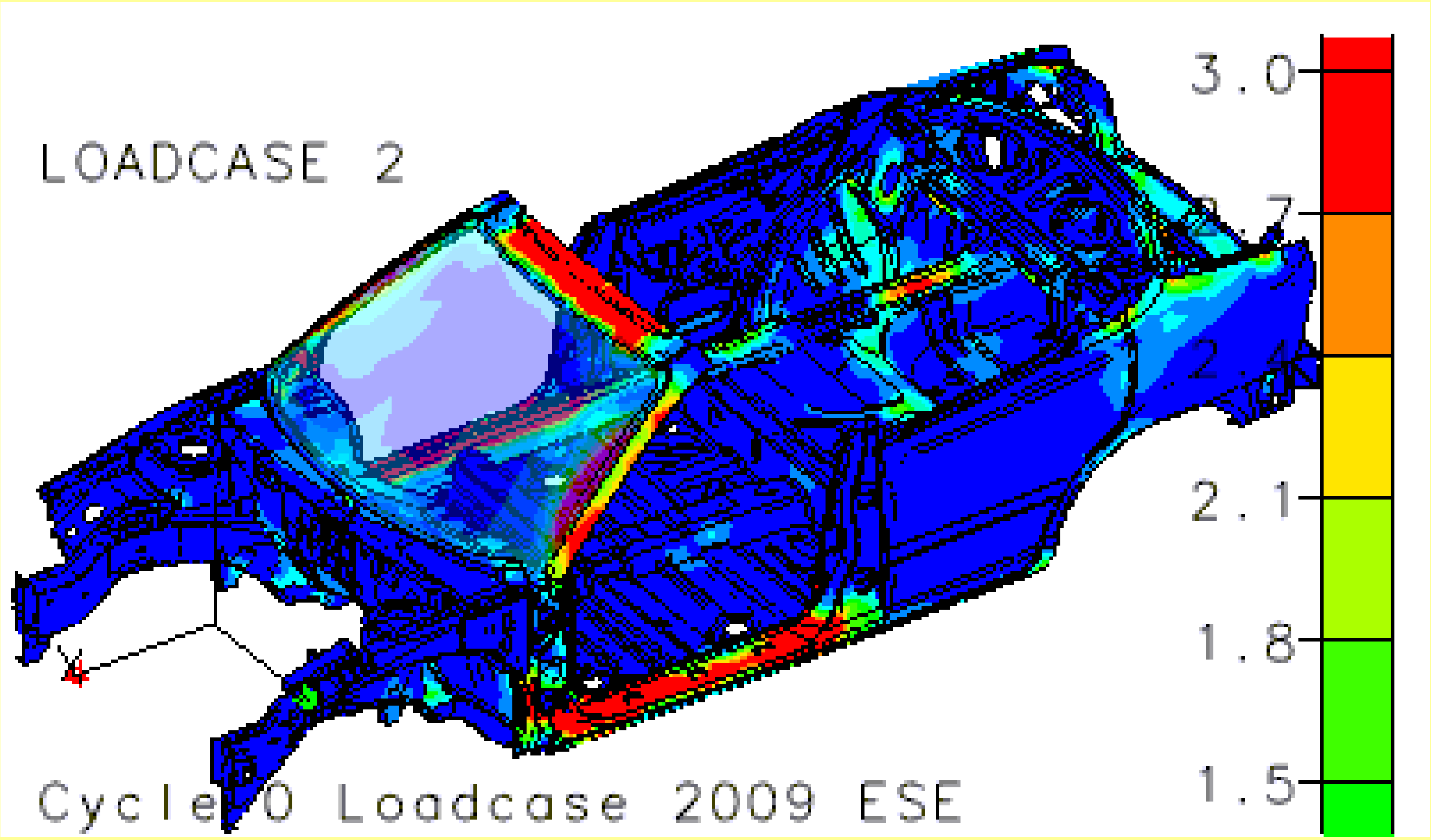




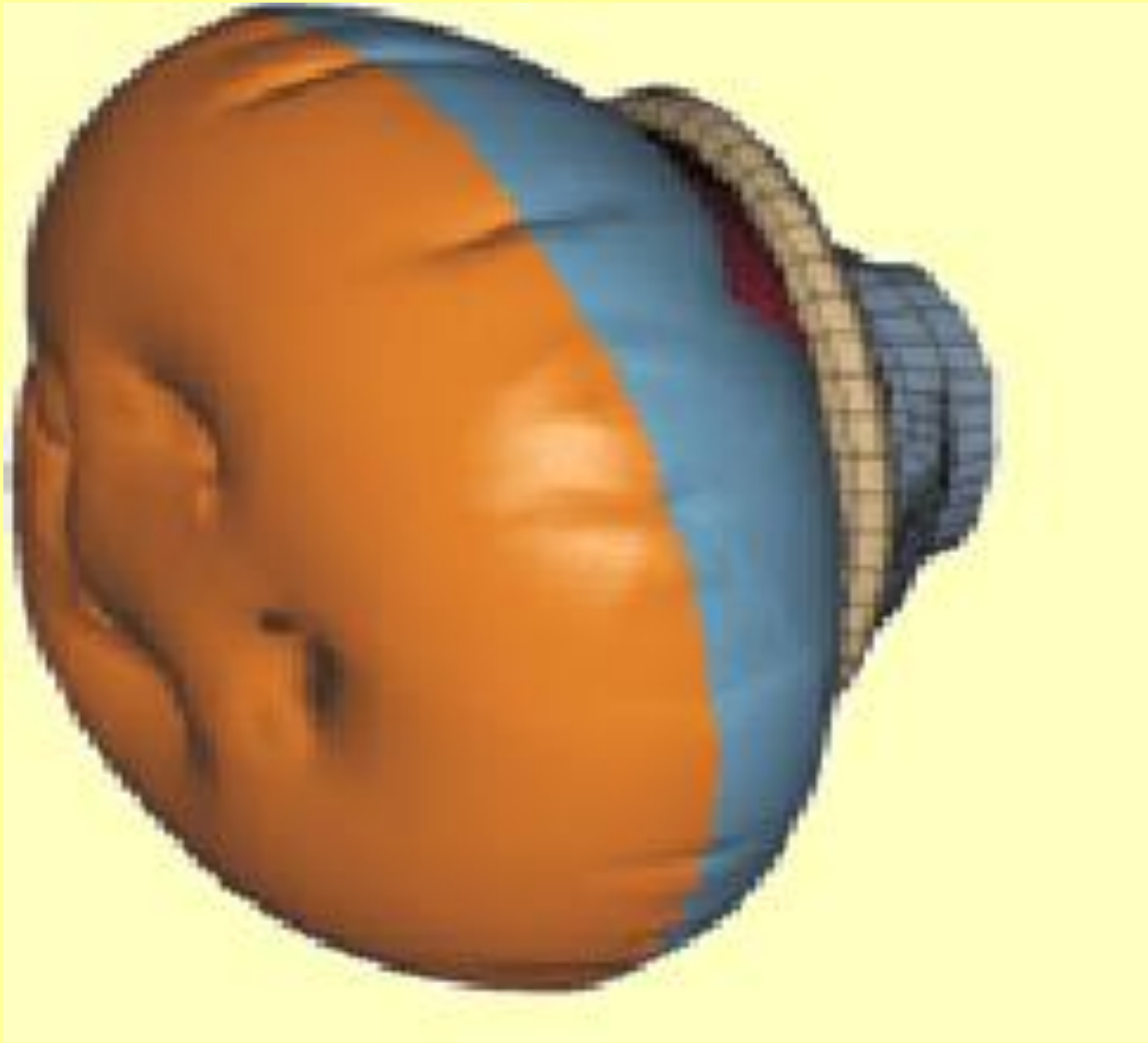


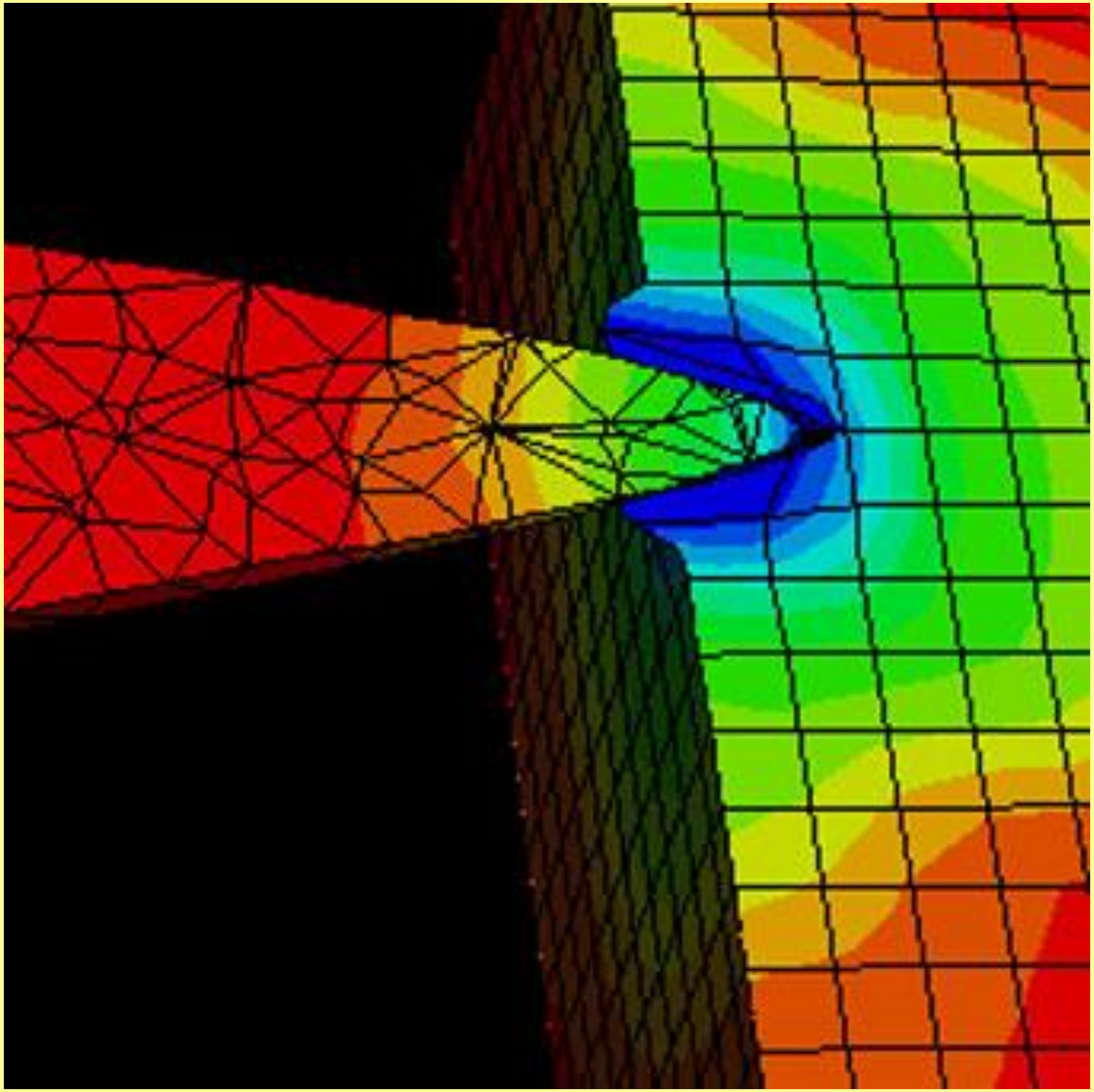

Simulated Part
Actual Part

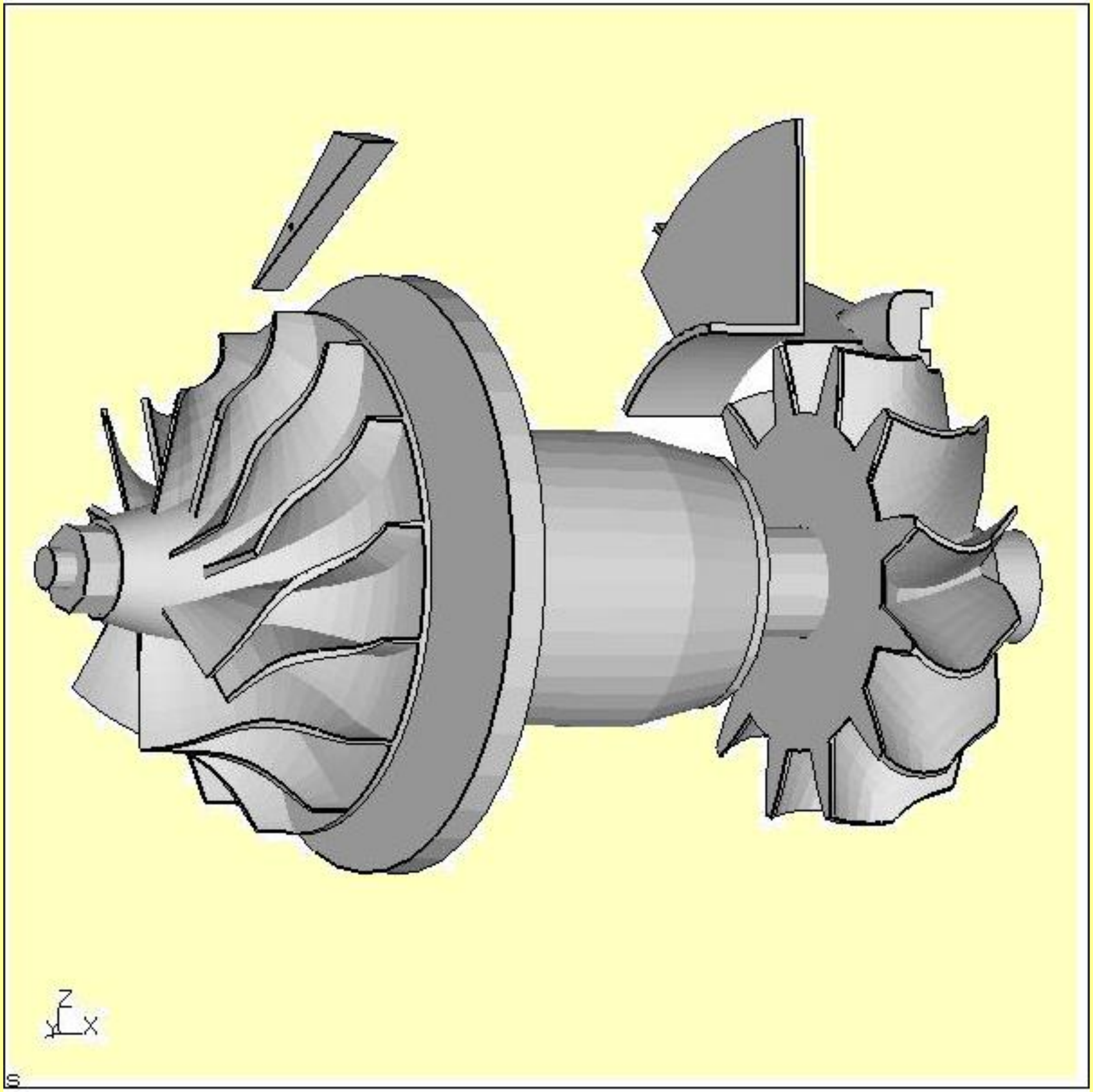



LOADCASE 2
Cycle 0 Loadcase 2009 ESE
3.0
2.1
1.8
1.5


YTI
2nd Mikkeli CompMechWorkshop'2007
CAD/FEA/CAE in Action
St.Petersburg State Polytechnical University
CompMechLab
Deformed Vehicle Views
Locked Pole






Z
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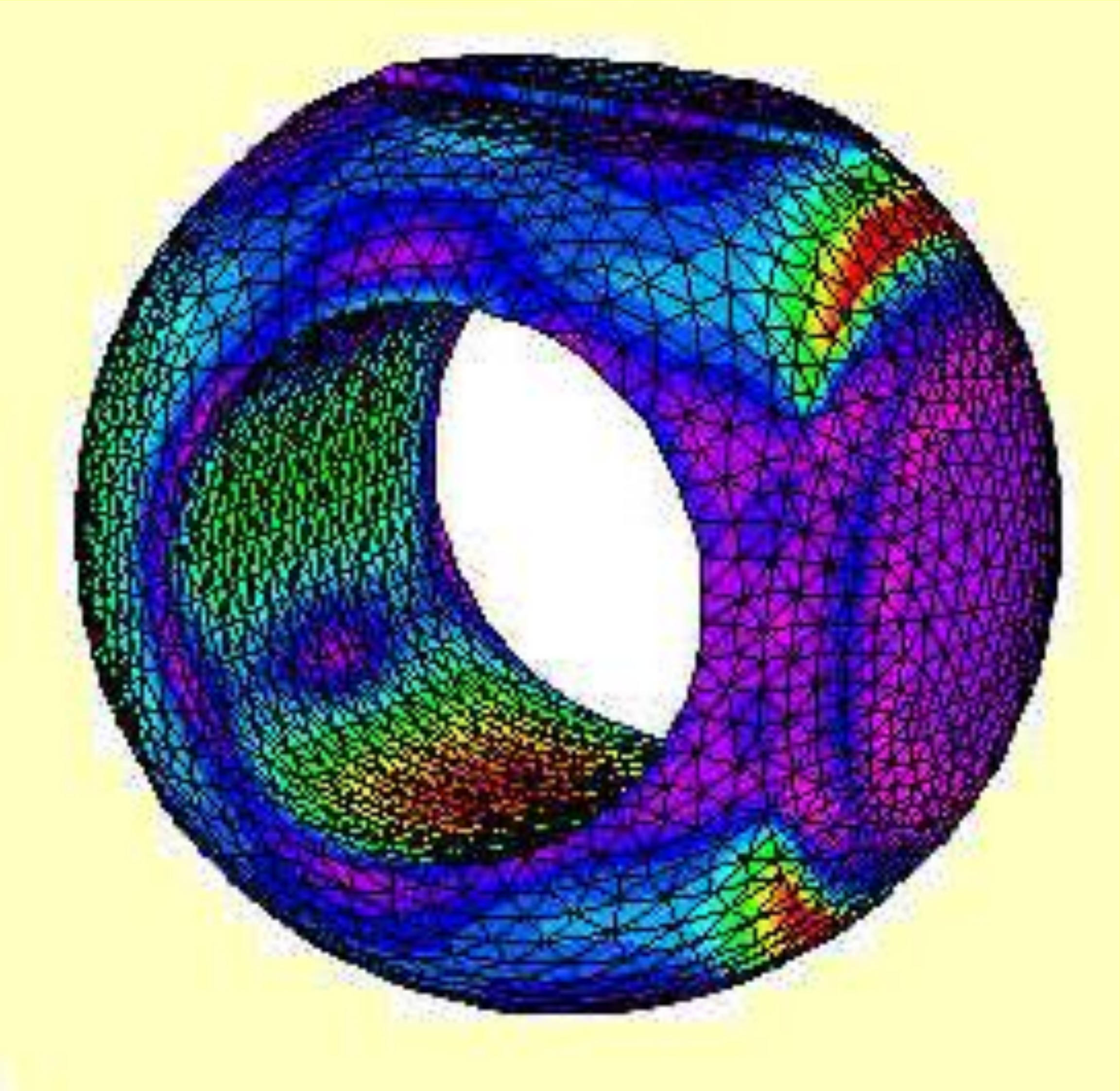
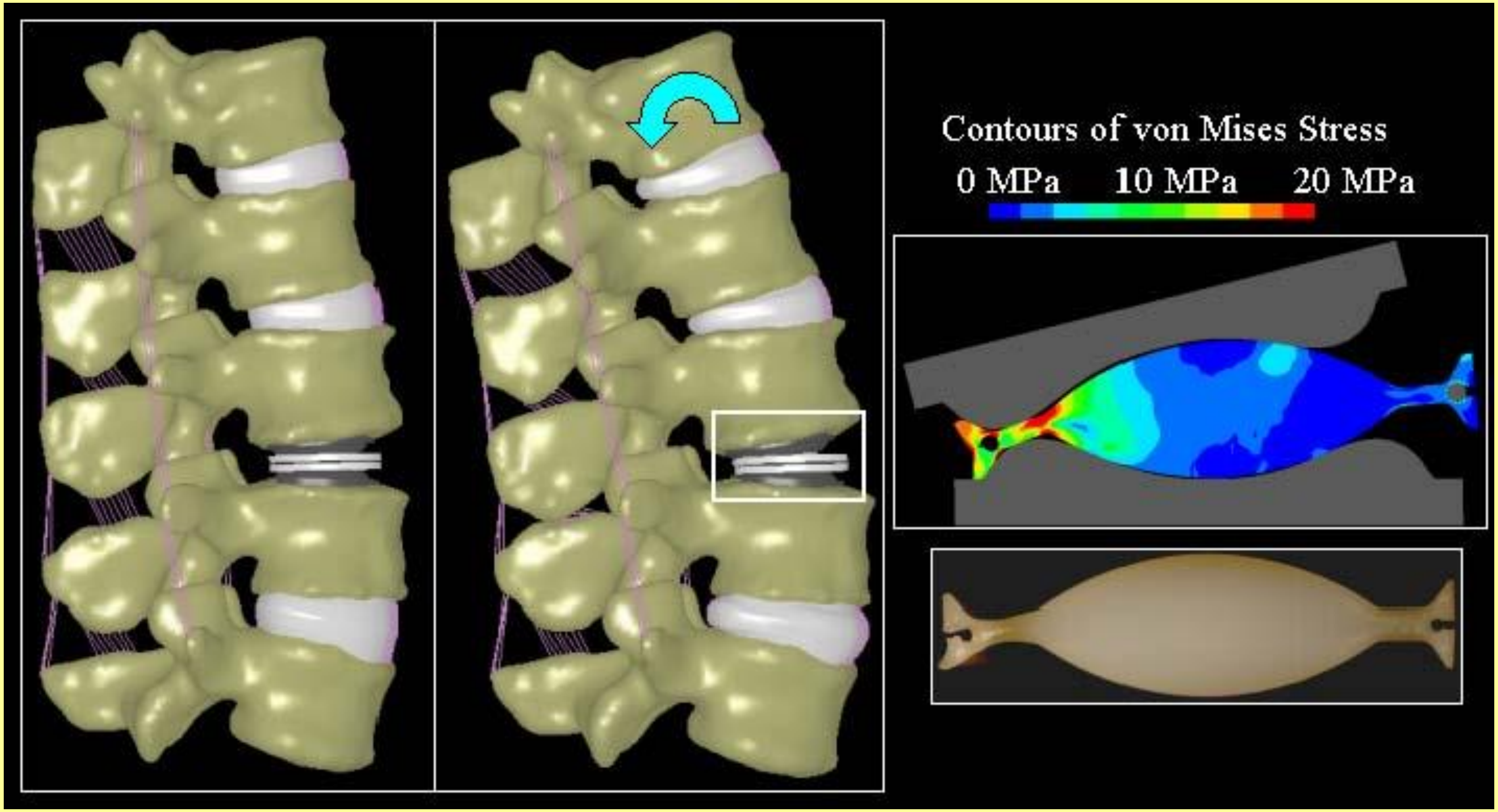
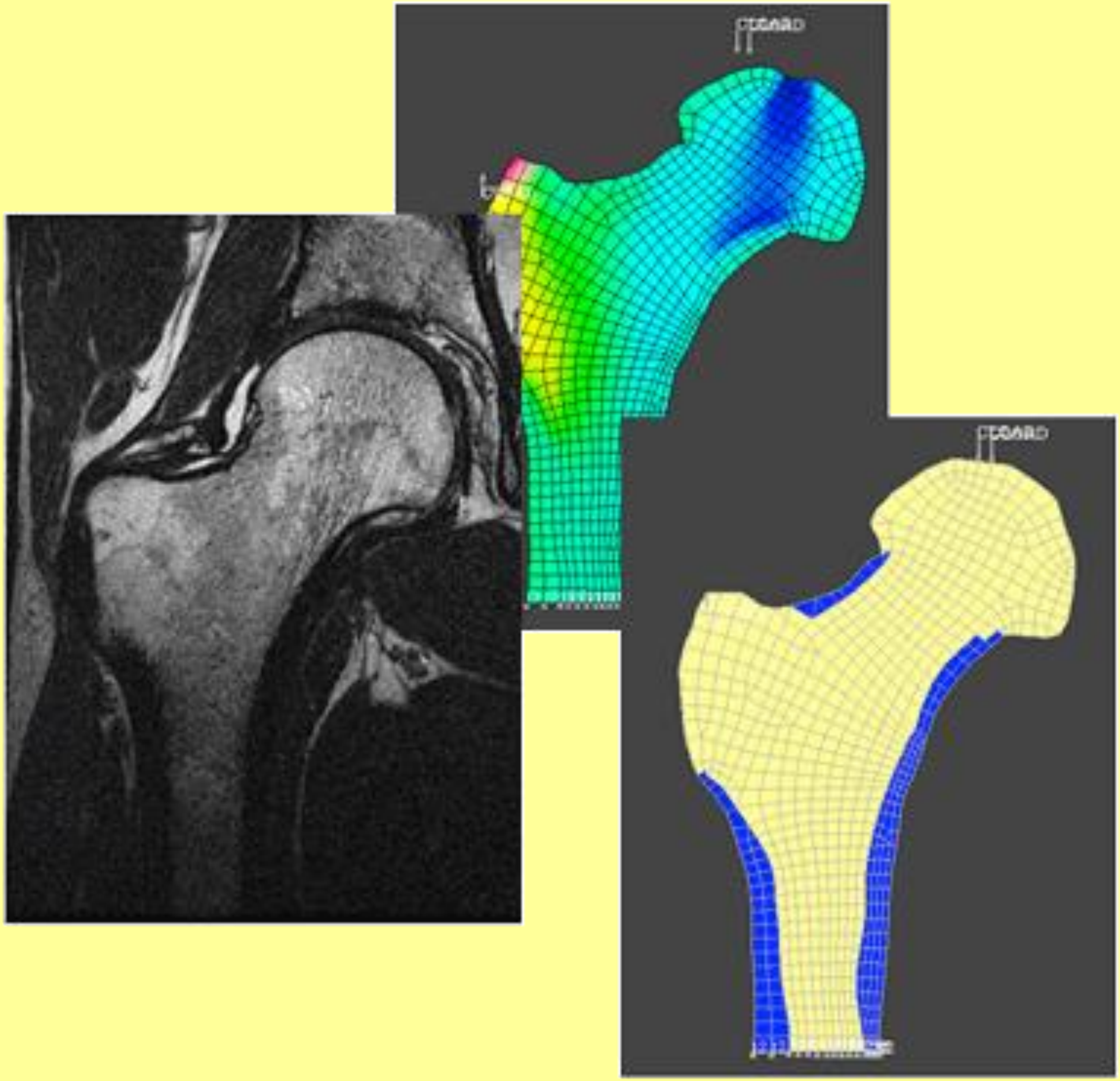
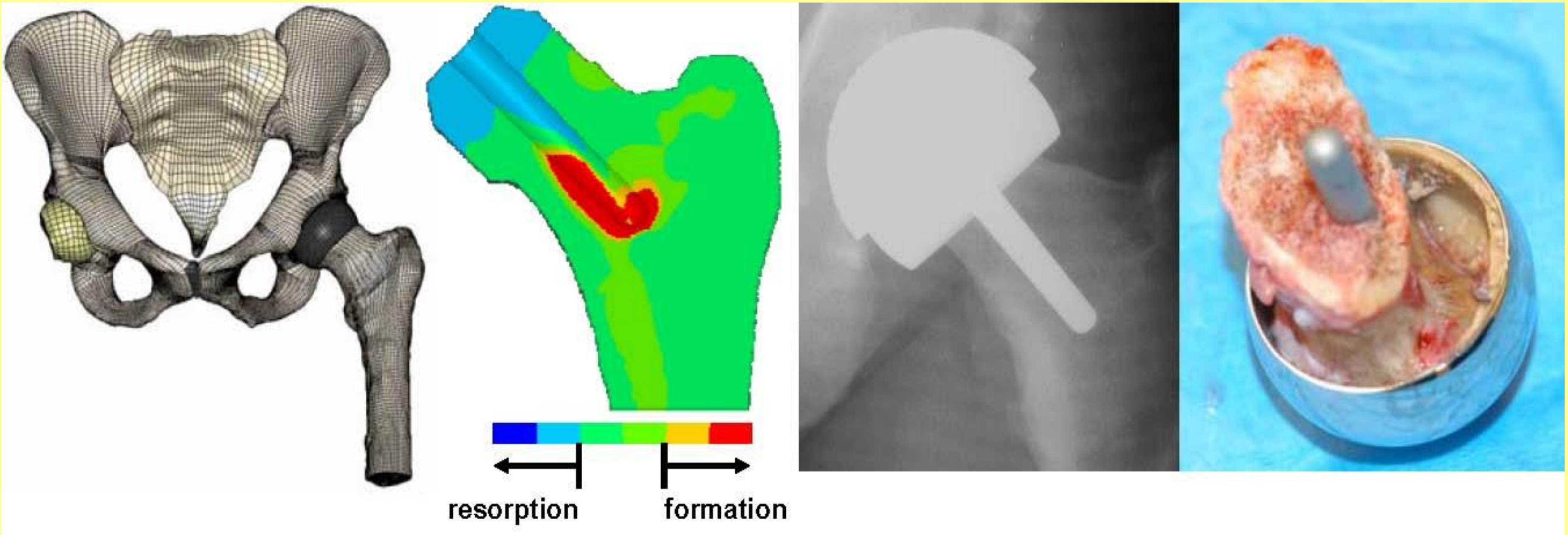



Contours of von Mises Stress
0 MPa
10 MPa
20 MPa




resorption
formation

FEM

FEA (finite element analysis), or FEM (finite element method), was primarily developed by engineers using physical reasoning and can trace much of its origin to matrix methods of structural analysis.

The finite element method is a computer aided mathematical technique that is used to obtain an approximate numerical solution to the fundamental differential and/or integral equations that predict the response of physical systems to external effects.

What is meant by external influence?

- ✦When a bar is subjected to an axial pull 'P' it elongates

- ✦When a metallic rod is heated its temperature rises

- ✦When a beam is subjected to an external harmonic excitation it vibrates

In the above examples the force ‘P’, or heat flux ‘q’ or harmonic excitation force constitute the “external influence” that causes the system to change.

The elongation, temperature rise or vibration represents the system’s response to the external influence.

Why FEM ?

Mathematical modeling to simulate physical happening

Any Physical System

Laws of physics

Mathematical modeling

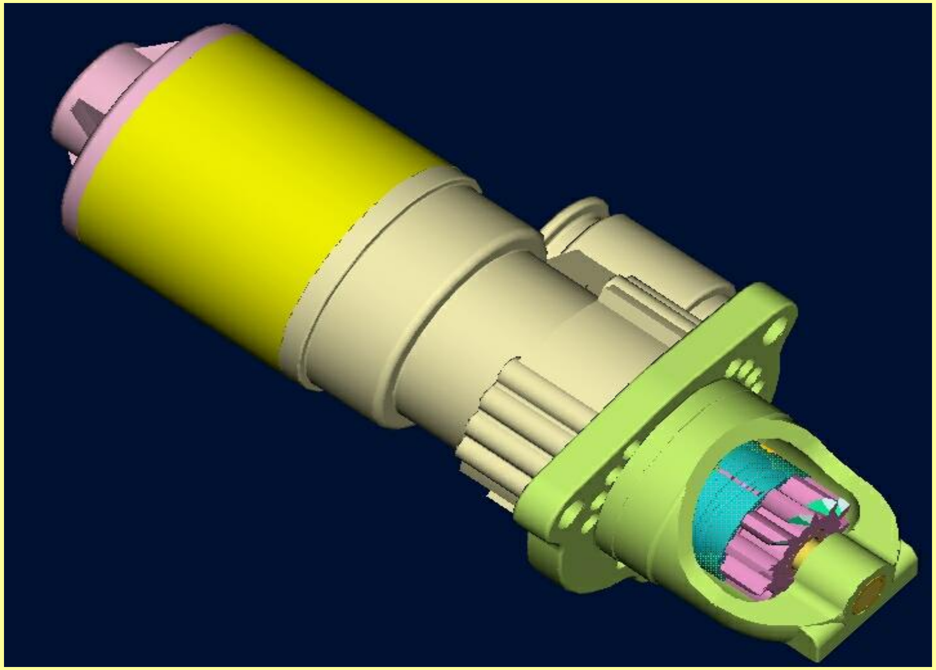
Solution

When FEM ?

Complex geometry

Complex loading

Complex material properties



Applications

- Structural Engineering
- Aerospace Engineering
- Automobile Engineering
- Thermal applications
- Acoustics
- Flow Problems
- Dynamics
- Metal Forming
- Medical & Dental applications
- Soil mechanics etc.

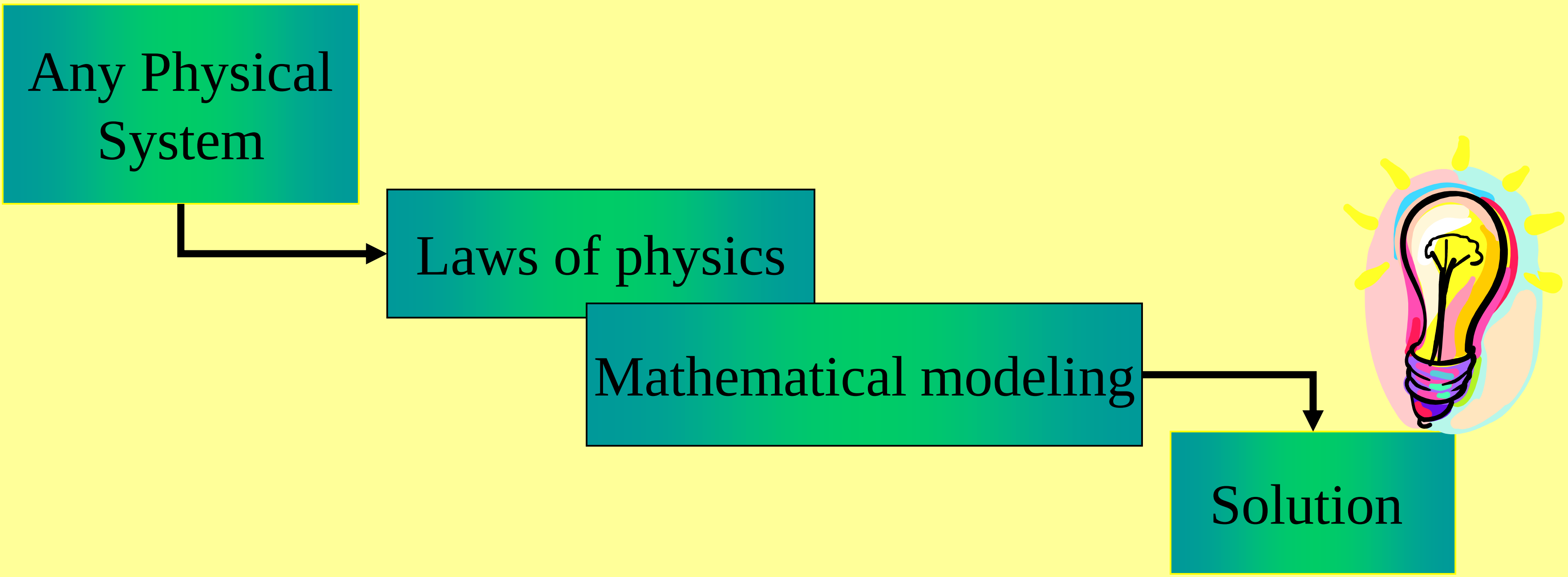
NUMERICAL SOLUTION TECHNIQUES

Weighted Residual Methods
- Collocation method
- Sub domain method
- Least squares method
- Galerkin method

Finite Difference Method
Rayleigh Ritz Technique
Finite Element Method
Boundary Element Method

FEM

Mathematical modeling to simulate physical happening



Mathematical modeling

Field Variable/ dependent variable Primary variable

Cross-sectional property

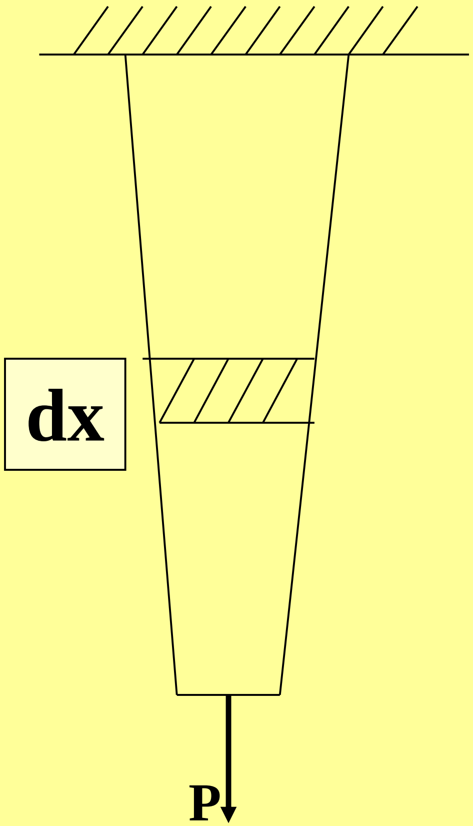
Governing Equation

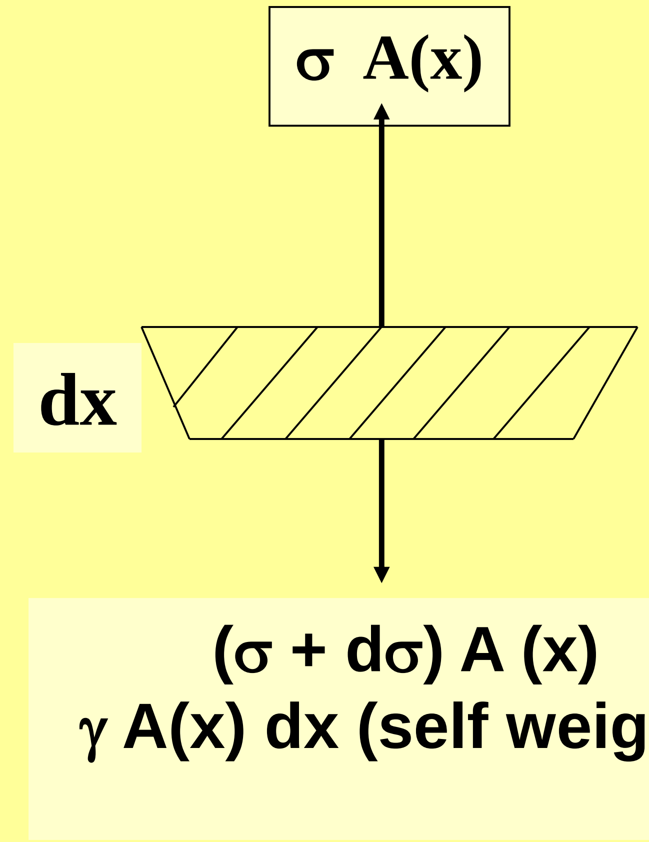
$$E\ A\ (d^2u/dx^2) + \gamma\ A = 0$$

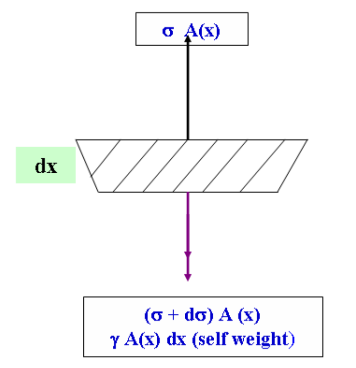
Material Properties

Independent variable or spatial co-ordinate

Example of a taper rod subjected a point load ‘P’ and its own self weight







For equilibrium $(\sigma + d\sigma)\,A(x) + \gamma\,A(x)\,dx - \sigma\,A(X) = 0$ --(1)

i.e) $d\sigma\,A(x) + \gamma\,A(x)\,dx = 0$ ---(2)

$$\sigma = E\varepsilon = E\frac{du}{dx} \rightarrow (3)$$

$A(x) = A0 - (A0-A1)\,x/l$

Where σ - stress, ∈ - strain & E - Young's Modulus

from continuum mechanics, $\in = du / dx$

(3) in (2) & dividing by dx.

$$E\frac{d(\sigma A(x))}{dx} + \gamma A(x) = 0$$

$$E\left(\frac{d\left[A(x)\frac{du}{dx}\right]}{dx}\right) + \gamma A(x) = 0 \rightarrow (4)$$

Governing Equation

For a bar of constant cross section

$$EA(x)\frac{d^2u}{dx^2} + \gamma A(x) = 0 \rightarrow (5)$$

$$\boxed{E\left(\frac{d\left[A(x)\dfrac{du}{dx}\right]}{dx}\right)+\gamma A(x)=0\rightarrow(4)}$$

Boundary conditions

1. $U(0) = 0$

2. $\left[EA(x)\dfrac{dU}{dx}\right]_{x=L} = P$

Variables:

- **Primary**

eg. Displacement, u

Temperature, T

- **Secondary**

eg. Force EA du/dx

Heat flux –KA dT/dx

Loads:

- Volume loads N/m^3 N/m

  eg. Self weight, udl

- Point loads N

Problems that could be solved by the FEM

1. Boundary Value Problems
2. Initial Value Problems
3. Eigen Value Problems

Boundary Value Problem (BVP)

A **boundary value problem** is one where the field variable (e.g., temperature or displacement) and possibly its derivatives are required to take on specified values on the boundary
(e.g.,

$$KA\, dT / dx = Q,$$

where K= Thermal conductivity,

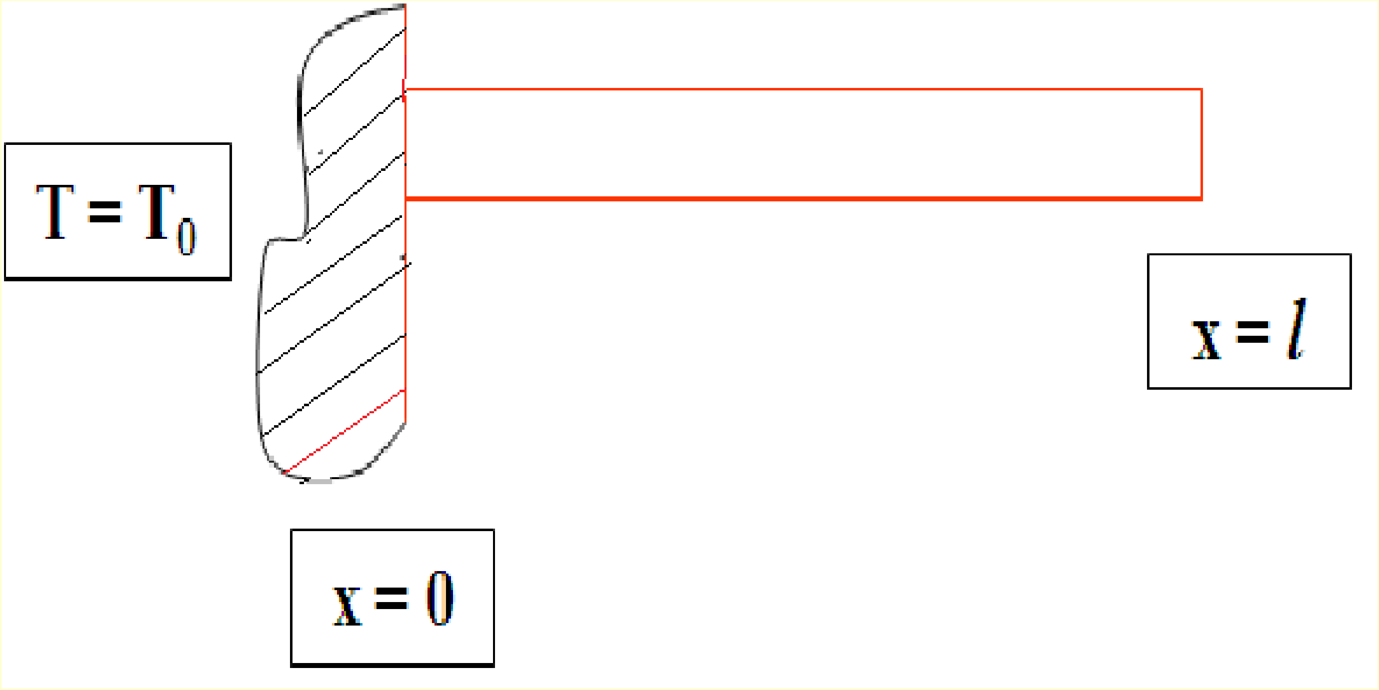
A = area of cross-section,

Q = Heat flux).

$$-\frac{d}{dx}\left[KA(x)\ \frac{dT(x)}{dx}\right] + hp\ [T(x)\ -\ T_{\infty}] = 0$$

Boundary conditions: @ x = 0, $\quad$ T = T_0

@ x = *l*, $\quad$ -KA (dt/dx) = 0



Initial Value Problem (IVP)

An Initial value problem is one where the field variable and possibly its derivatives are specified initially (i.e., at time t=0). These are generally time dependent problems.

Examples include

Unsteady heat conduction

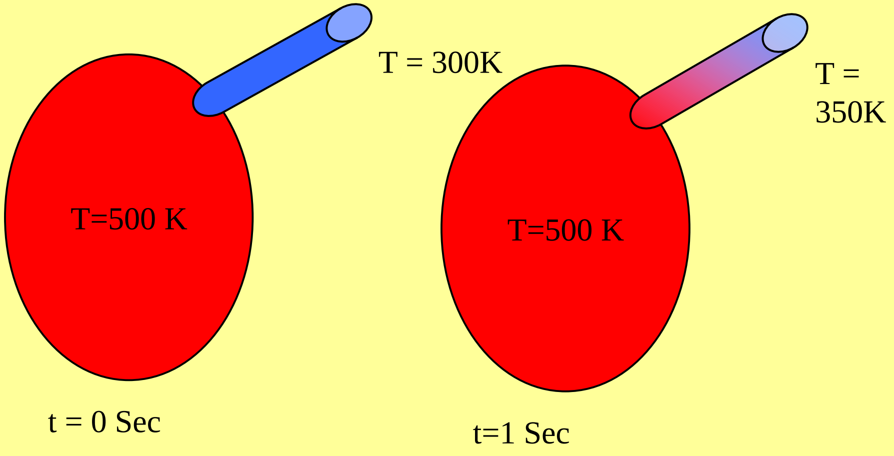
Dynamic problems

Initial conditions: @ time t = 0

i) $du/dt = C_0$

where Velocity = du /dt

ii) displacement $u = a_0$

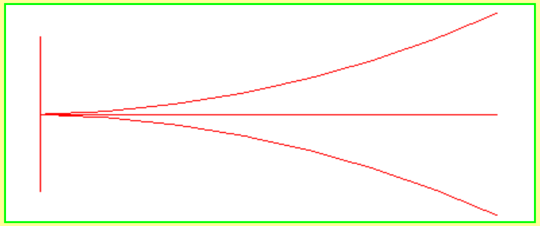


Eigen Value Problem (EVP)

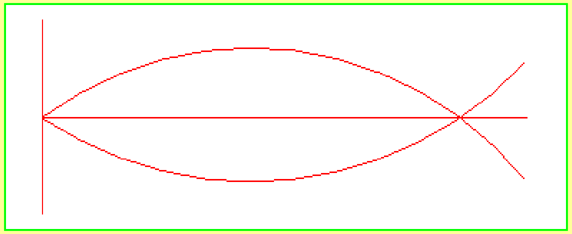
An eigen value problem is one where the problem is defined by a homogeneous differential equation that is one where the right hand side is zero. An important class of eigen value problems is the ‘Vibration of Beams” or continuous systems.

Eigen Value Problem (EVP)

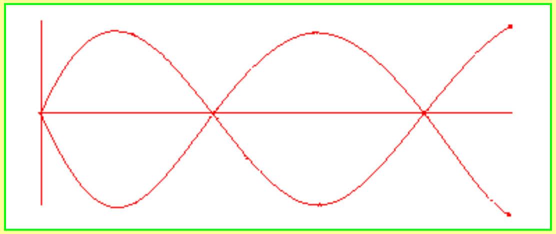
First mode shape



Second mode shape



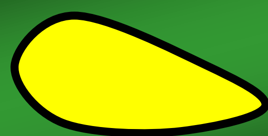
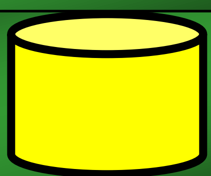
Third mode shape



DIMENSIONALITY

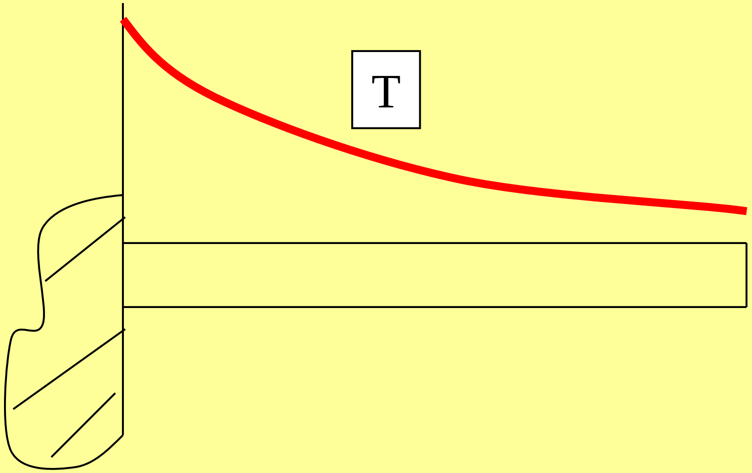
Physical problems can be classified into

(i) I dimensional
(ii) II dimensional
(iii) III dimensional problems.

Domain	Geometry	Boundary
1D	Line	Points
2D	Area 	Curves
3D	Volume 	Area

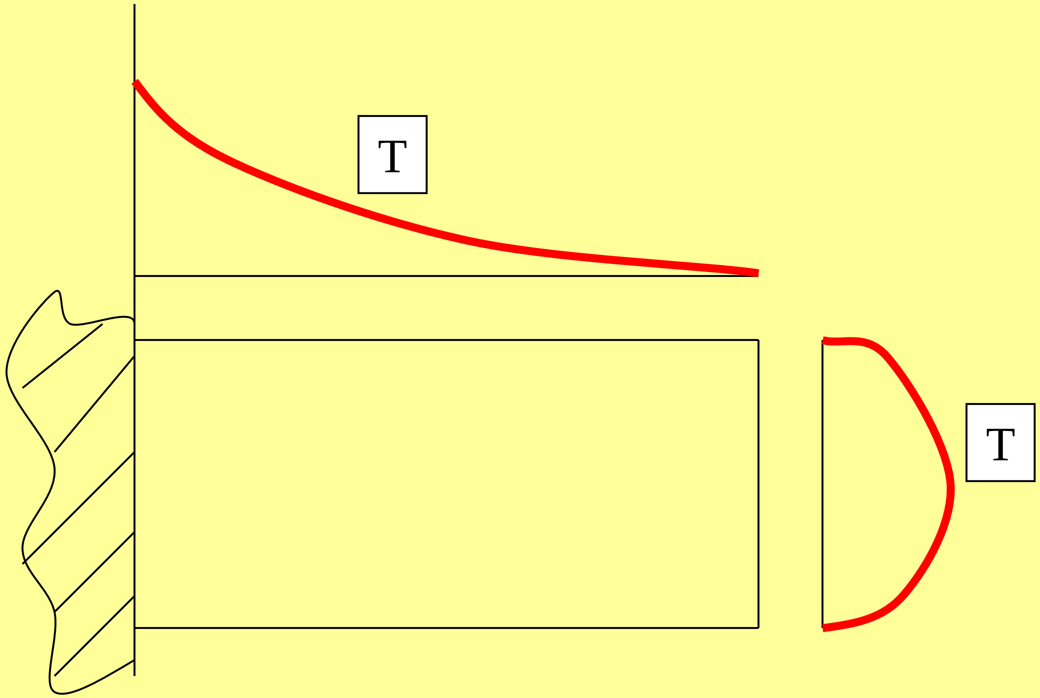
I-D PROBLEMS:-

When the geometry, material properties and field variables such as displacement, temperature, pressure etc can be described in terms of only one spatial co-ordinate we can go in for one-dimensional modeling



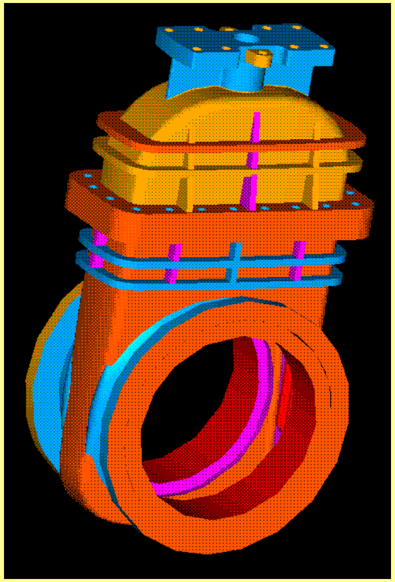
2D PROBLEMS:-

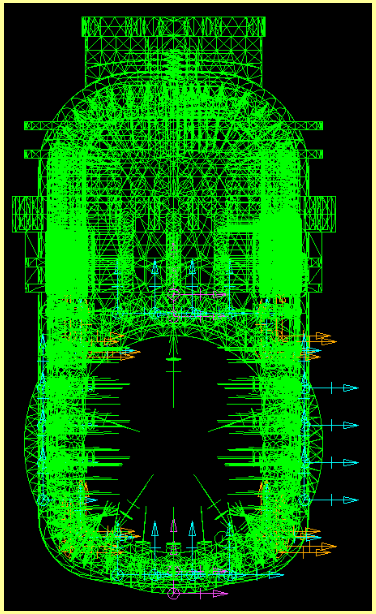
When the geometry and other parameters are described in terms of two independent co-ordinates we go in for two-dimensional modeling.



3D PROBLEMS:-

If the geometry, material properties and other parameters of the body can be described by three independent spatial co-ordinates, we can discretize the body using 3 dimensional modeling.





Exact and approximate solutions:

- **An exact solution satisfies the differential equation at every point in the domain and the boundary conditions on the boundary**
- **An approximate solution satisfies the boundary conditions completely and as closely as possible the differential equation**

$$E\left(\frac{d\left[A(x)\frac{du}{dx}\right]}{dx}\right)+\gamma A(x)=0$$

Boundary conditions

1. $U(0) = 0$

2. $\left[EA(x)\dfrac{dU}{dx}\right]_{x=L} = P$

$$E\left(\frac{d\left[A(x)\dfrac{d\bar{u}}{dx}\right]}{dx}\right)+\gamma A(x)=R$$

R – RESIDUE

$\bar{u}$ - approximate solution

$u_{ex} - \bar{u}$ = Error in solution

NUMERICAL SOLUTION OF BVPs

(i) Choose a trial solution $\overline{U}(x)$ for $U(x)$

(ii) Select a criterion for minimising the error

$\overline{U}(x)$ can be a trigonometric function such as Asinx

or a logarithmic function log x

or a hyperbolic function

or polynomial functions

$$\overline{U}(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$

$$\overline{U}(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

$$U(x) = a_0 \phi_0 (x) + a_1 \phi_1 (x) + \ldots + a_n \phi_n (x)$$

$$\phi_i = x^i$$

1. **Methods of weighted residuals** (WRM) which are applicable when the governing equations are differential equations.

2. **Ritz variational method** which is applicable when the governing equations are variational (integral) equations with an associated quadratic functional.

The WRM criteria seek to minimise the error involved in not satisfying the governing differential equations.

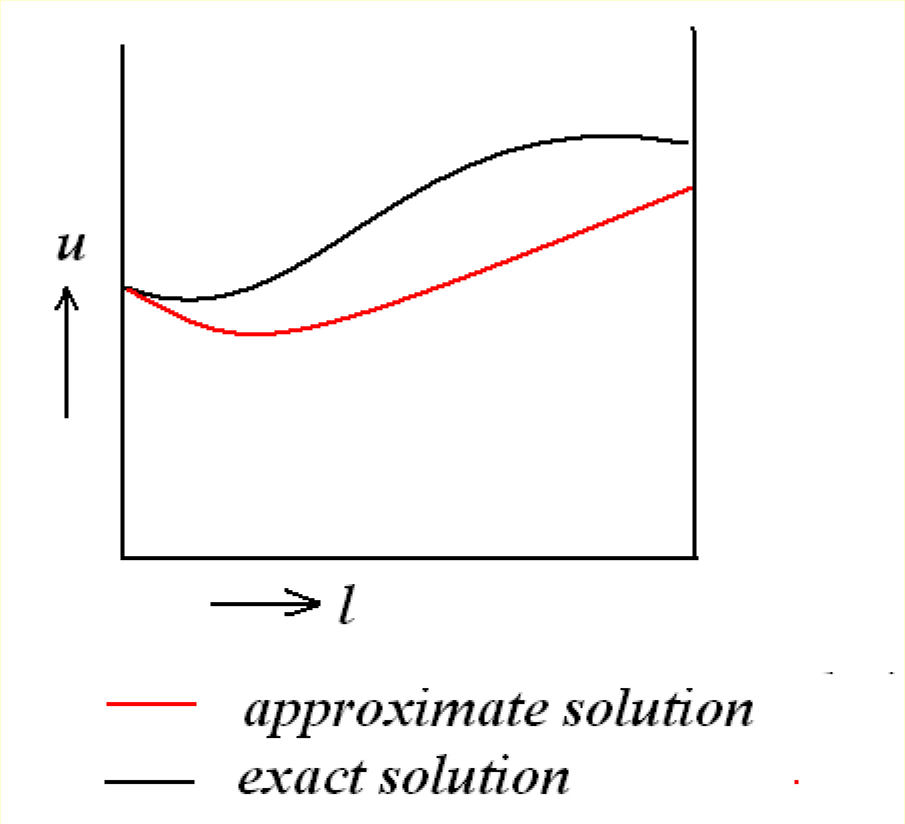
The most popular methods are

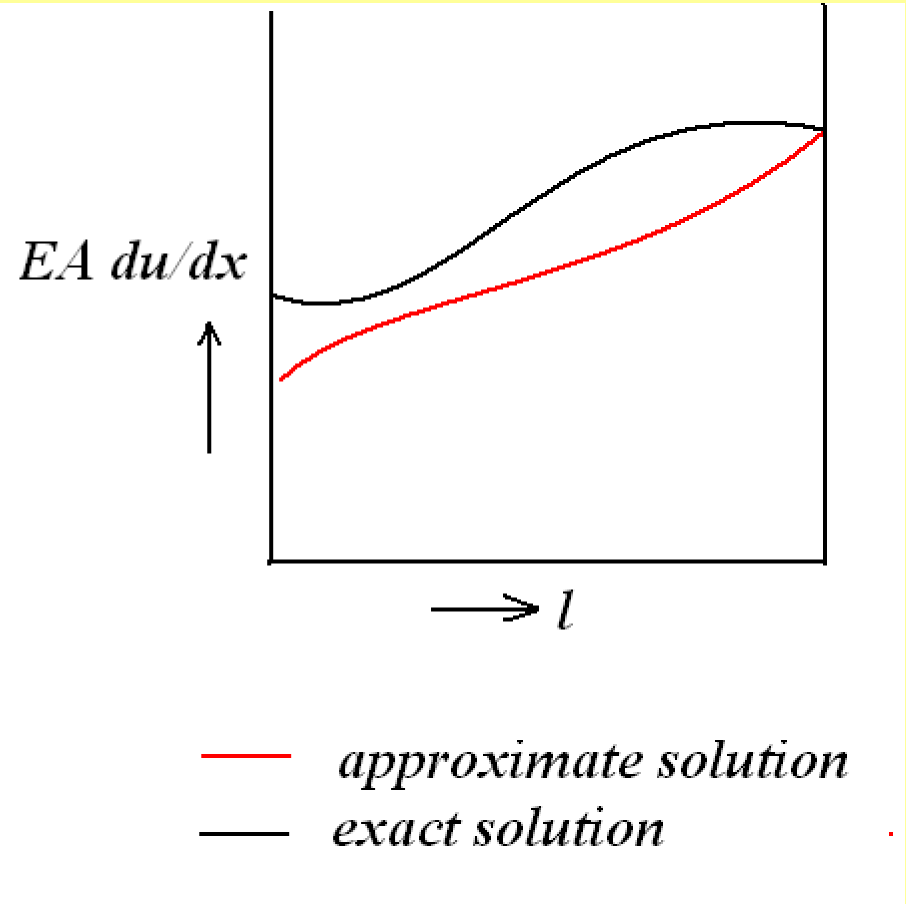
(i) The Collocation method.

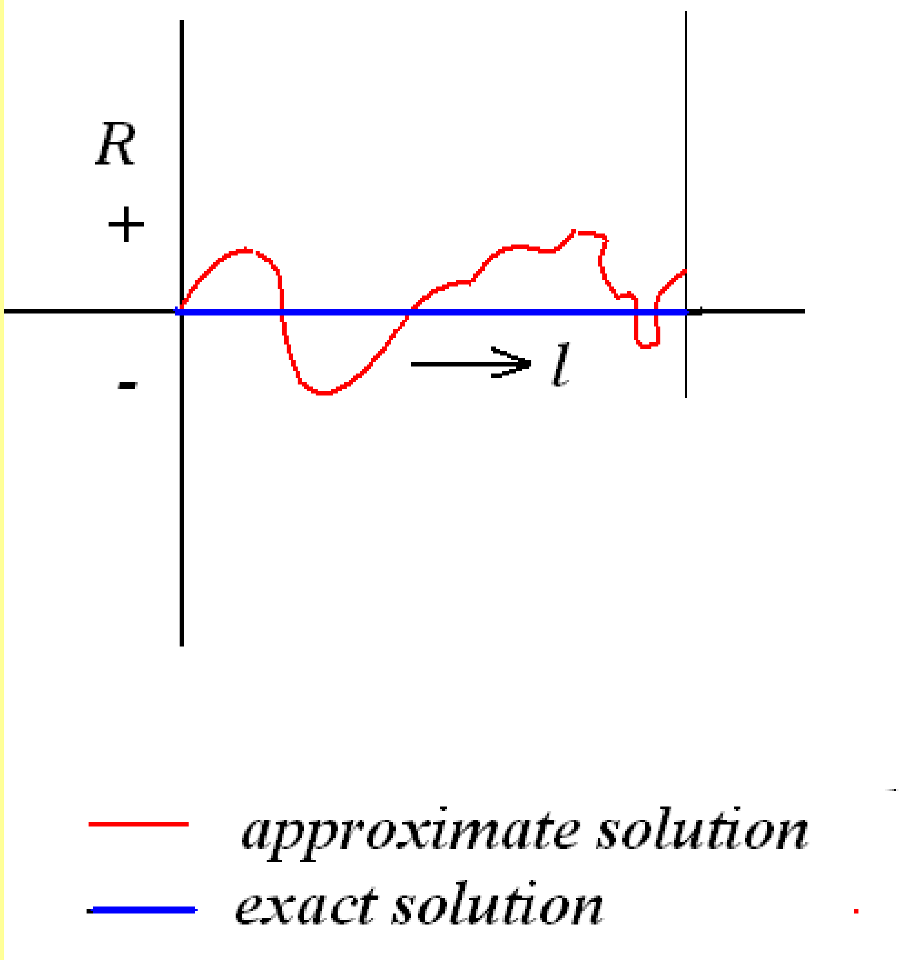
(ii) The Sub-Domain method

(iii) The Least squares method.

(iv) The Galerkin method.


u
l
approximate solution
exact solution


EA du/dx
l
approximate solution
exact solution



R
+
-
l
approximate solution
exact solution

COLLOCATION METHOD

For each undetermined coefficient a_i, choose a point x_i in the domain and at each such point called as collocation point force the residual to be exactly zero

$$R(x_1) = 0$$

$$R(x_2) = 0$$

$$\cdots\cdots\cdots\cdots$$

ie. $$R(x_n) = 0$$

The collocation points may be located anywhere on the boundary or in the domain.

THE SUB-DOMAIN METHOD

For each undetermined parameter choose an interval Δx, in the domain. Then force average of the residual in each interval to be zero.

……………………

……………………

$$\frac{1}{\Delta x_1} \int_{\Delta x_1} R(x)\, dx = 0$$

$$\frac{1}{\Delta x_2} \int_{\Delta x_2} R(x)\, dx = 0$$

$$\frac{1}{\Delta x_n} \int_{\Delta x_n} R(x)\, dx = 0$$

LEAST SQUARES TECHNIQUE:

In this method we minimize with respect to each undetermined coefficient the integral of the square of the residue over the entire domain

$$\partial / \partial \ a_1 \int_1^2 R^2 (x) \, dx = 0$$

$$\int_1^2 R(x)(\partial R/\partial \bar{a}_1) \, dx = 0$$

THE GALERKIN METHOD

For each undetermined parameter we require that a weighted average of R(x) over the entire domain be zero. The weighting functions are the trial functions associated with the generalised coefficients

$$\int_1^2 R(x)\,\phi_i(x)\,dx = 0$$

GENERAL WRM

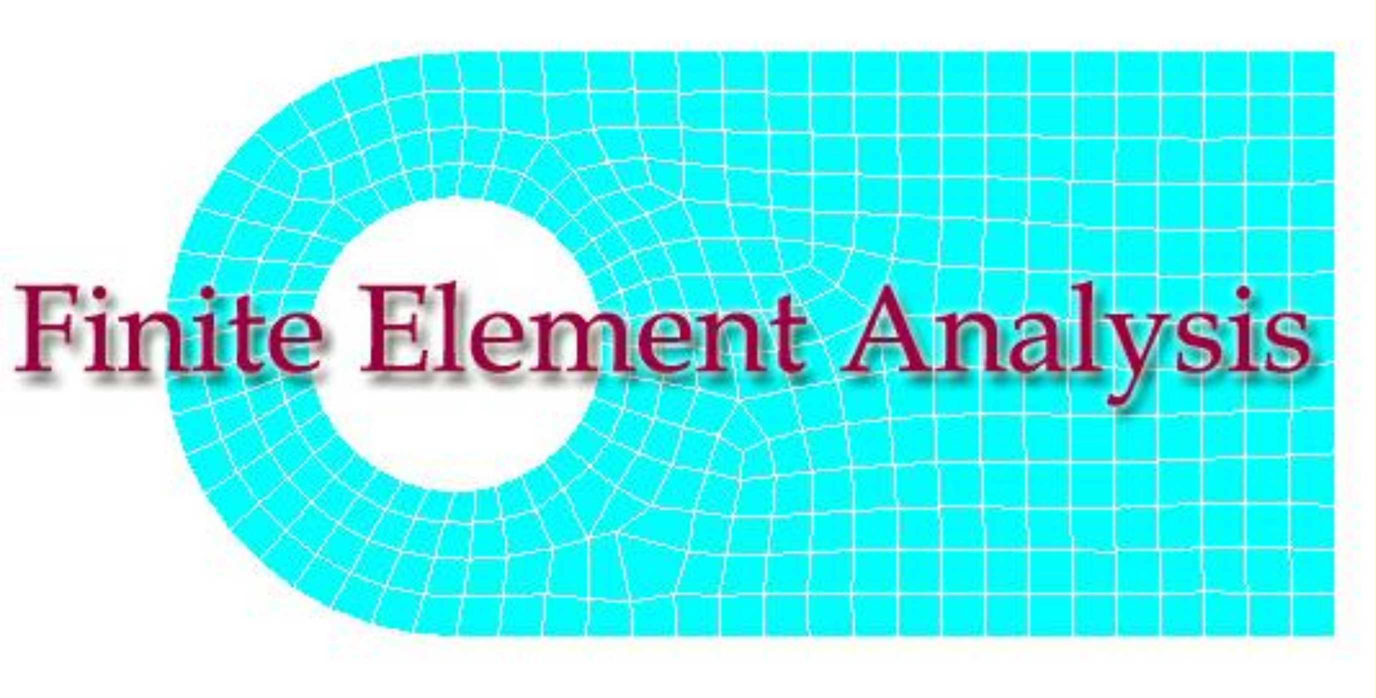
$$\int_{\Omega} R(X)\, w_i\,(x)\, dx = 0 \qquad i = 1, 2, \ldots, n$$

(i) The Collocation method - **dirac delta function**

(ii) The Sub-Domain method - **Unity**

(iii) The Least squares method - **Residue**

(iv) The Galerkin method – **coefficient of the undetermined coefficients in the trial solution**


Finite Element Analysis

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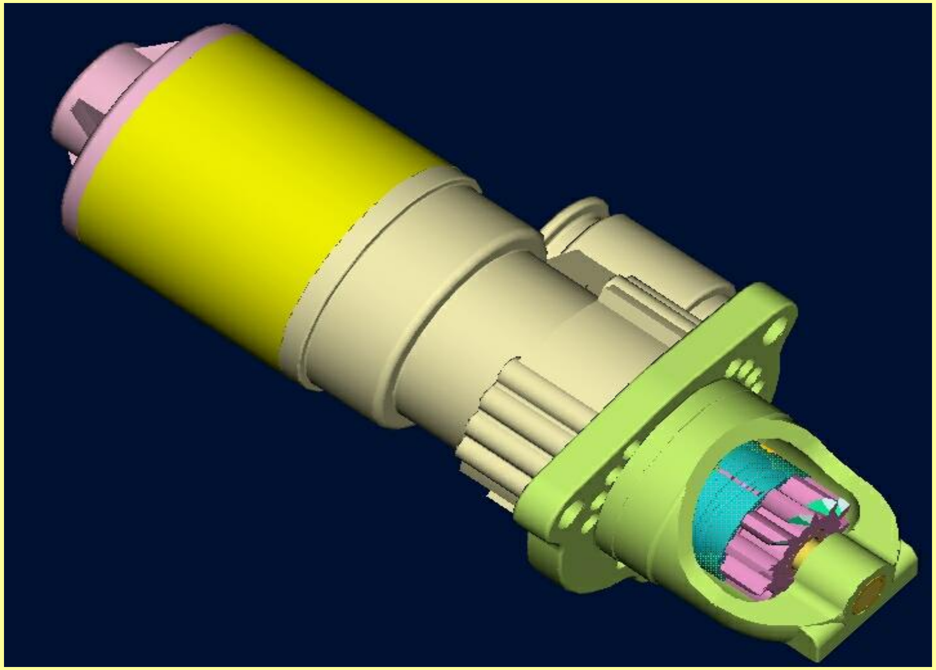
When FEM ?

Complex geometry

Complex loading

Complex material properties



FEM

Mathematical modeling to simulate physical happening

Any Physical System

Laws of physics

Mathematical modeling

Solution

$$E\left(\frac{\delta\left[A(\xi)\quad\frac{\delta v}{\delta\xi}\right]}{\delta\xi}\right)+\gamma\gamma A(\xi) \qquad 0 \rightarrow (\quad 4 \;)$$

Boundary conditions

1. $u(0) = 0$

2. $$\mathrm{EA(x)}\,\frac{\mathrm{du}}{dx}\Big|_{x=L} = \mathrm{P}$$

Variables:

- **Primary**

  eg. Displacement, u

  Temperature, T

- **Secondary**

eg. Force EA du/dx

Heat flux –KA dT/dx

Moment – EI (d^2w/dx^2)

BOUNDARY CONDITIONS:

- **Essential/ Geometric/ Dirichlet Boundary conditions**
- **Natural/ Force/ Neumann Boundary conditions**

BOUNDARY CONDITIONS CAN BE OF THE FOLLOWING TWO TYPES

- HOMOGENEOUS eg. $u(0)= 0$
- NON-HOMOGENEOUS eg. $T(0)=80$

Loads:

- **Volume loads** N/m^3 N/m

  eg. Self weight, udl

- **Point loads** N

Exact and approximate solutions:

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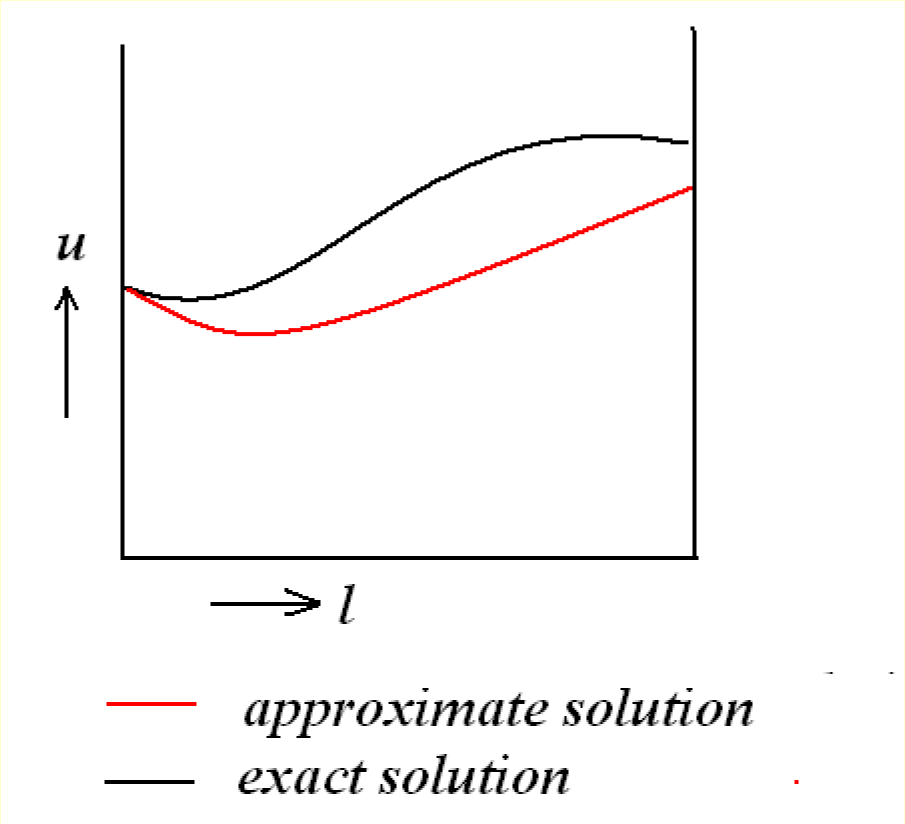
$u_{ex} - \bar{u}$ = Error in solution


u
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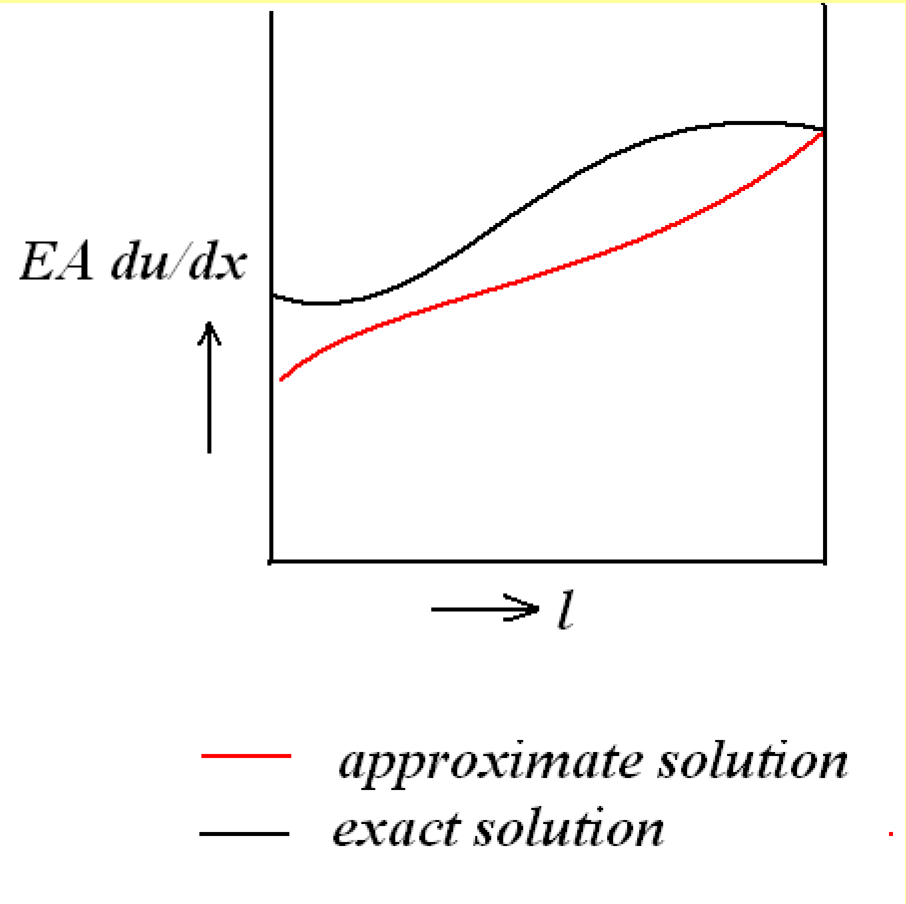
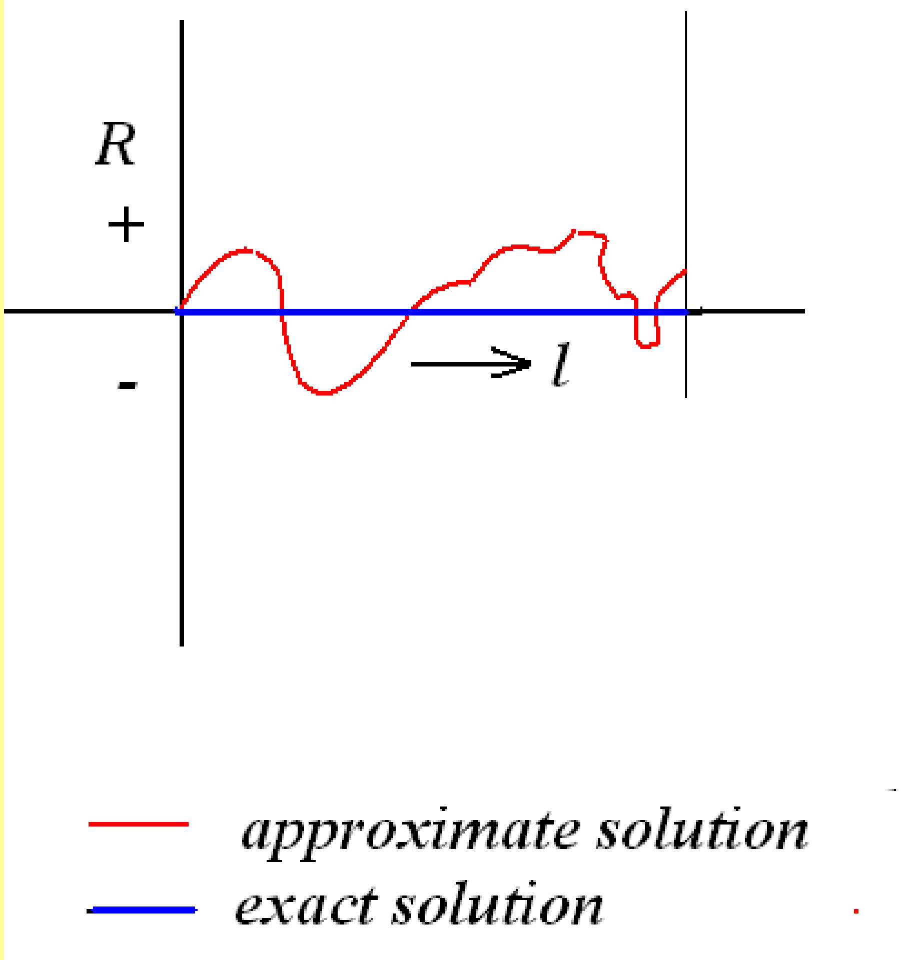
approximate solution
exact solution


EA du/dx
l
approximate solution
exact solution



R
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approximate solution
exact solution

NUMERICAL SOLUTION OF BVPs

(i) Choose a trial solution $\bar{u}(x)$ for $u(x)$

(ii) Select a criterion for minimising the error

$\bar{u}(x)$ can be

a trigonometric function such as Asinx

or a logarithmic function log x

or a hyperbolic function

or polynomial functions

$$\bar{u}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\bar{u}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

$$\boxed{\varphi_i = x^i}$$

$$u(x) = a_0 \varphi_0(x) + a_1 \varphi_1(x) + \ldots + a_n \varphi_n(x)$$

The WRM criteria seek to minimise the error involved in not satisfying the governing differential equations. The most popular criteria are

(i) The Collocation method.

(ii) The Sub-Domain method

(iii) The Least squares method.

(iv) The Galerkin method.

CONSTRUCTION OF A TRIAL SOLUTION

We know that any function f(x) can be expanded in a power series as

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

Thus the function f(x) can be written as a sum of series of functions with appropriate constants. Similarly the approximate or trial solution is sought in the form

$$u(x) = \mathrm{a}_0\, \varphi_0\,(\mathrm{x}) + \mathrm{a}_1\, \varphi_1\,(\mathrm{x}) + \ldots + \mathrm{a}_\mathrm{n}\, \varphi_\mathrm{n}(\mathrm{x})$$

$$u(x) = a_0\,\phi_0(x) + a_1\,\phi_1(x) + \ldots + a_n\,\phi_n(x)$$

$\phi_i(x)$ - trial functions or basis functions

a_i - undetermined constants or generalised co-ordinates

Generalised Co-ordinates approach

1. **Methods of weighted residuals** (WRM) which are applicable when the governing equations are differential equations.

2. **Ritz variational method** (RVM) which is applicable when the governing equations are variational (integral) equations with an associated quadratic functional.

ILLUSTRATIVE PROBLEM

Consider the equation

$$\frac{d}{dx}\left[\mathrm{x}\,\frac{\mathrm{du}}{\mathrm{dx}}\right]=\frac{2}{\mathrm{x}^{2}}$$

in the domain 1 < x < 2

with B.Cs as i) u(1) = 2 and

ii) $$\left[-\mathrm{x}\,\frac{\mathrm{du}}{\mathrm{dx}}\right]_{X=2}=\frac{1}{2}$$

$$\left[-x\,\frac{du}{dx}\right]_{X=2}=\frac{1}{2}$$

$$\left[\frac{du}{dx}\right]_{X=2}=-\frac{1}{4}$$

$$\left[-x\,\frac{du}{dx}\right]$$

Flux/ secondary variable

Let $\bar{u}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

BC (i) $\rightarrow$ $\bar{u}(1) = a_0 + a_1 + a_2 + a_3 = 2$

or $a_0 = 2 - a_1 x - a_2 - a_3$ ---------- (1)

BC (ii) $\rightarrow$ $\left[-x \frac{d\bar{u}}{dx}\right]_{x=2} = -2(a_1 + 4a_2 + 12a_3) = \frac{1}{2}$

$$a_1 = -\frac{1}{4} - 4a_2 - 12a_3$$ ---------- (2)

Substituting for a_1 and a_2 in the expression for $\overline{U}(x)$,we have

$$\bar{u}(x) = 2 - \frac{1}{4}(x-1) + a_2 (x-1)(x-3) + a_3 (x-1)(x^2 + x - 11)$$

$$= \phi_0 + \bar{a}_1 \phi_1 (x) + \bar{a}_2 \phi_2 (x)$$

where $$\phi_0 = 2 - \frac{1}{4}(x-1)$$

$$\phi_1 = (x - 1)(x - 3)$$

$$\phi_2 = (x - 1)(x^2 + x - 11)$$

$$\bar{u}(x) = 2 - \frac{1}{4}(x - 1) + a_2 (x - 1)(x - 3) + a_3 (x - 1)(x^2 + x - 11)$$

It can be easily seen that the above trial function satisfies the conditions imposed on the boundary. Thus the construction of trial function is over.

WRM APPLICATION

Consider the equation

$$\frac{d}{dx}\left(x\frac{\mathrm{du}}{\mathrm{dx}}\right)=\frac{2}{x^2}$$

or

$$\frac{d}{dx}\left(x\frac{\mathrm{du}}{\mathrm{dx}}\right)-\frac{2}{x^2}=0$$

Substituting the trial solution $\bar{u}$ (x) for u(x), this equation is unlikely to be satisfied.
i.e., the RHS is a non-zero function, R(x)

i.e. R(x) = $\dfrac{d}{dx}\left(x\,\dfrac{du}{dx}\right) - \dfrac{2}{x^2} \neq 0$

This is called as the 'Residual' and is a measure of the error involved in not satisfying the Governing equation.

R(x) = - $\dfrac{1}{4} + 4(x - 1)\, a_2 + 3(3x^2 - 4)a_3 - \dfrac{2}{x^2}$

COLLOCATION METHOD

For each undetermined coefficient $\bar{a}_i$ choose a point x_i in the domain and at each such point x_i force the residual to be exactly zero

i.e,

$$
\begin{aligned}
R(x_1) &= 0 \\
R(x_2) &= 0 \\
&\ldots\ldots\ldots\ldots \\
R(x_n) &= 0
\end{aligned}
$$

The chosen points are called collocation points. They may be located any were on the boundary or in the domain. For the present problem we have 2 undetermined coefficients a_2 & a_3.

Choose $x_1 = 4/3$ & $x_2 = 5/3$

Substituting in the expression for R(x), we get

$$\frac{4}{3}a_2 + 4a_3 = \frac{11}{8}$$

$$\frac{8}{3}a_2 + 13a_3 = \frac{97}{100}$$

Solving the simultaneous equations

$$a_2 = 2.0993 \;\; \& \; a_3 = -0.356$$

therefore,

$$\bar{u}(x) = 2 - \frac{1}{4}(x-1) + 2.0993\,(x-1)(x-3) -$$

$$0.356\,(x-1)(x^2 + x - 11)$$

THE SUB-DOMAIN METHOD

For each undetermined parameter a_i, choose an interval Δx_i in the domain. Then force average of the residual in each interval to be zero.

$$\frac{1}{\Delta x_1} \int_{\Delta x_1} R(x)\, dx = 0$$

$$\frac{1}{\Delta x_2} \int_{\Delta x_2} R(x)\, dx = 0$$

$$\frac{1}{\Delta x_n} \int_{\Delta x_n} R(x)\, dx = 0$$

which yields a system of n residual equations which can be solved for a_i. The intervals Δx_i are called the ‘sub domains.’ . They may be chosen in any fashion.

Taking

$$\Delta x_1 \quad 1 < x < 1.5 \quad \&$$

$$\Delta x_2 \quad 1.5 < x < 2$$

$$\frac{1}{0.5}\int_{1.5}^{2} R(x)\,\mathrm{dx} = 0$$

$$\frac{1}{0.5}\int_{1}^{1.5} R(x)\,\mathrm{dx} = 0$$

we get
$$a_2 = 2.5417$$
$$a_3 = -0.4529$$

$$\bar{u}(x) = 2 - \frac{1}{4}(\mathrm{x}-1) + 2.5417\,(\mathrm{x}-1)(\mathrm{x}-3) -$$
$$0.4529\,(\mathrm{x}-1)(\mathrm{x}^2 + x - 11)$$

LEAST SQUARES TECHNIQUE

In this method, we minimize with respect to each undetermined coefficient the integral of the square of the residue over the entire domain

$$\partial / \partial \ a_i \int_1^2 R^2 (x) \, dx = 0$$

$$\int_1^2 2 R(x)(\partial R/\partial a_i) \ dx = 0$$

$$\int_1^2 2\,R(x)(\partial\,\mathrm{R}/\partial\,\mathrm{a}_2\,)\quad \mathrm{dx} \quad = \; 0$$

$$\int_1^2 2\,R(x)(\partial\,\mathrm{R}/\partial\,\mathrm{a}_3)\quad \mathrm{dx} \quad = \; 0$$

$$a_2 \;=\; 2.3155$$

$$a_3 \;= -\,0.3816$$

$$\bar{u}(x) = 2 - \frac{1}{4}(\mathrm{x} - 1) + 2.3155\;(\mathrm{x} - 1)\,(\mathrm{x} - 3) -$$

$$0.3816\;(\mathrm{x} - 1)\,(\mathrm{x}^2 + x - 11)$$

THE GALERKIN METHOD

For each parameter a_i, we require that a weighted average of R(x) over the entire domain be zero. The weighting functions are the trial functions $\phi_i(x)$ associated with a_i

$$\int_1^2 R(x)\,\phi_i(x)\,dx = 0$$

$$\bar{u}(x) = 2 - \frac{1}{4}(\mathrm{x} - 1) + \mathrm{a}_2\,(\mathrm{x} - 1)\,(\mathrm{x} - 3) + \mathrm{a}_3\,(\mathrm{x} - 1)\,(\mathrm{x}^2 + x - 11)$$

$$\int_1^2 R(x)\,(\mathrm{x} - 1)(\mathrm{x} - 3)\,\mathrm{dx} = 0$$

$$\int_1^2 R(x)\,(\mathrm{x} - 1)(\mathrm{x}^2 + \mathrm{x} - 11)\,\mathrm{dx} = 0$$

This yields

$$a_2 = 2.3178$$

$$a_3 = -0.3477$$

$$\bar{U}(x) = 2 - \frac{1}{4}(\mathrm{x} - 1) + 2.3178\,(\mathrm{x} - 1)\,(\mathrm{x} - 3) - 0.3477\,(\mathrm{x} - 1)\,(\mathrm{x}^2 + x - 11)$$

$$\int_{\Omega} R(x)\, w_i(x)\, dx = 0 \qquad i = 1, 2, \ldots, n$$

i) The Collocation method - **dirac delta function**

ii) The Sub-Domain method - **Unity**

iii) The Least squares method - **Residue**

iv) The Galerkin method – **coefficient of the undetermined coefficients in the trial solution**

$$\int_{\Omega} R(x)\, w_{\mathrm{i}}\,(\mathrm{x})\,\mathrm{dx} = 0 \qquad \mathrm{i} = 1, 2, \ldots, \mathrm{n}$$

The Collocation method - **dirac delta function**

$$\int_{\Omega} R(x)\,\delta\,\mathrm{dx} = 0$$

$$\delta = (\mathrm{x} - \mathrm{x}_0)$$

δ is zero every where except at x = x_o

$$\int_{\Omega} R(x)\,1\,\mathrm{dx} = 0 \qquad \mathrm{i} = 1, 2, \ldots, \mathrm{n}$$

The Sub-Domain method - **Unity**

$$\int_{\Omega} R(x)\,\mathrm{R}\,\mathrm{dx} = 0$$

The Least squares method - **Residue**

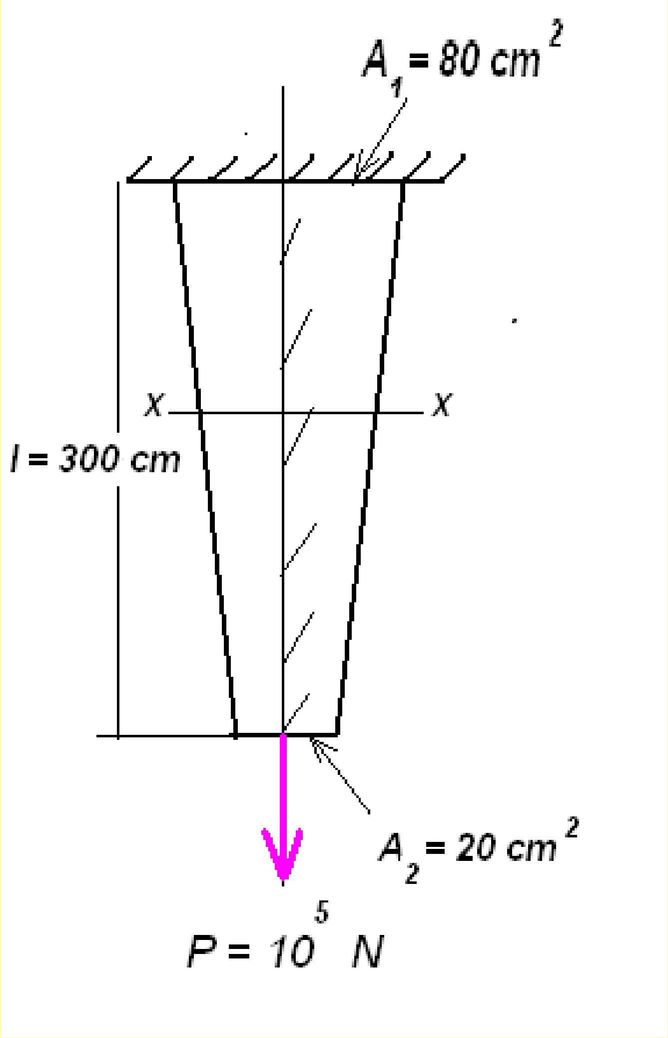
$$\int_{\Omega} R(x)\,\phi_{\mathrm{i}}\,(\mathrm{x})\,\mathrm{dx} = 0$$

The Galerkin method – **coefficient of the undetermined coefficients in the trial solution**

Examples of One-Dimensional BVPs

1. Elastic deformation of a bar

A tapered circular bar made of steel is suspended vertically with the larger end rigidly clamped and the smaller end acted on by a pull of 10^5 N. The areas at the larger and smaller ends are 80 cm^2 and 20 cm^2, respectively. The length of the bar is 3m. The bar weighs 0.075 N/cc. Young's modulus of the bar material is $E = 2 \times 10^7$ N/ cm^2. Obtain an approximate expression for the deformation of the rod.



$A(x) = A_1 – (A_1 - A_2)\ x/l$

ie. $A(x) = 80 – (80-20)x/300$

$= (80 – 0.2x)$

γ = 0.075 N/cm3

E = 2 x 107 N/cm2

Governing equation of the problem is

$$\frac{d}{dx}\left[\mathrm{EA(x)}\,\frac{\mathrm{du}}{\mathrm{dx}}\right] + \gamma\mathrm{A(x)} = 0 \qquad 0 < \mathrm{x} < \mathrm{L} \qquad \text{------- } 1$$

With the boundary conditions

$$u(0) = 0 \text{ and } \left[\mathrm{EA(x)}\,\frac{\mathrm{du}}{dx}\right]_{x=L} = \mathrm{P} \qquad \text{-------- } 2$$

Given

P $=10^5$ N $\quad\gamma$ = 0.075 N/cm^3

E = 2 x 10^7 N/cm^2 L = 300 cm and

A(x) = (80 - 0.2 x) cm

Step 1 Choice of Trial Function

Let $\bar{u}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ ----- 3

Applying the B.Cs (1) and (2) we have

$$a_0 = 0 \quad \text{and} \quad a_1 = 2.5 \times 10^{-4} - 600\, a_2 - 27 \times 10^4\, a_3$$

The trial solution takes the form

$$u(x) = x\,[\,2.5 \times 10^{-4} - (600 - x)\, a_2 - (27 \times 10^4 - x^2)\, a_3\,]$$

Step II Optimising Criterion using the Collocation Method

The residual at any point is given by

$$R(x) = 2 \text{ x } 10^{7} \text{ x } [-0.5 \text{ x } 10^{-4} + a_2 (280 - 0.8x) + a_3 (5.4 \text{ x } 10^{4} + 480 \text{ x} - 1.8 \text{ x}^{2}) + 3\, x\, 10^{-7} - 0.75 \text{ x } 10^{-7}\, x\,]$$

Choosing the two points x_1 =100 cm & x_2 =200 cm and forcing $R(x_1)$ & $R(x_2)$ to take zero values, we arrive at a simultaneous equation for a_2 & $a_3$and the solution of which turns out to be

$$a_2 = 0.21846 \times 10^{-6}$$

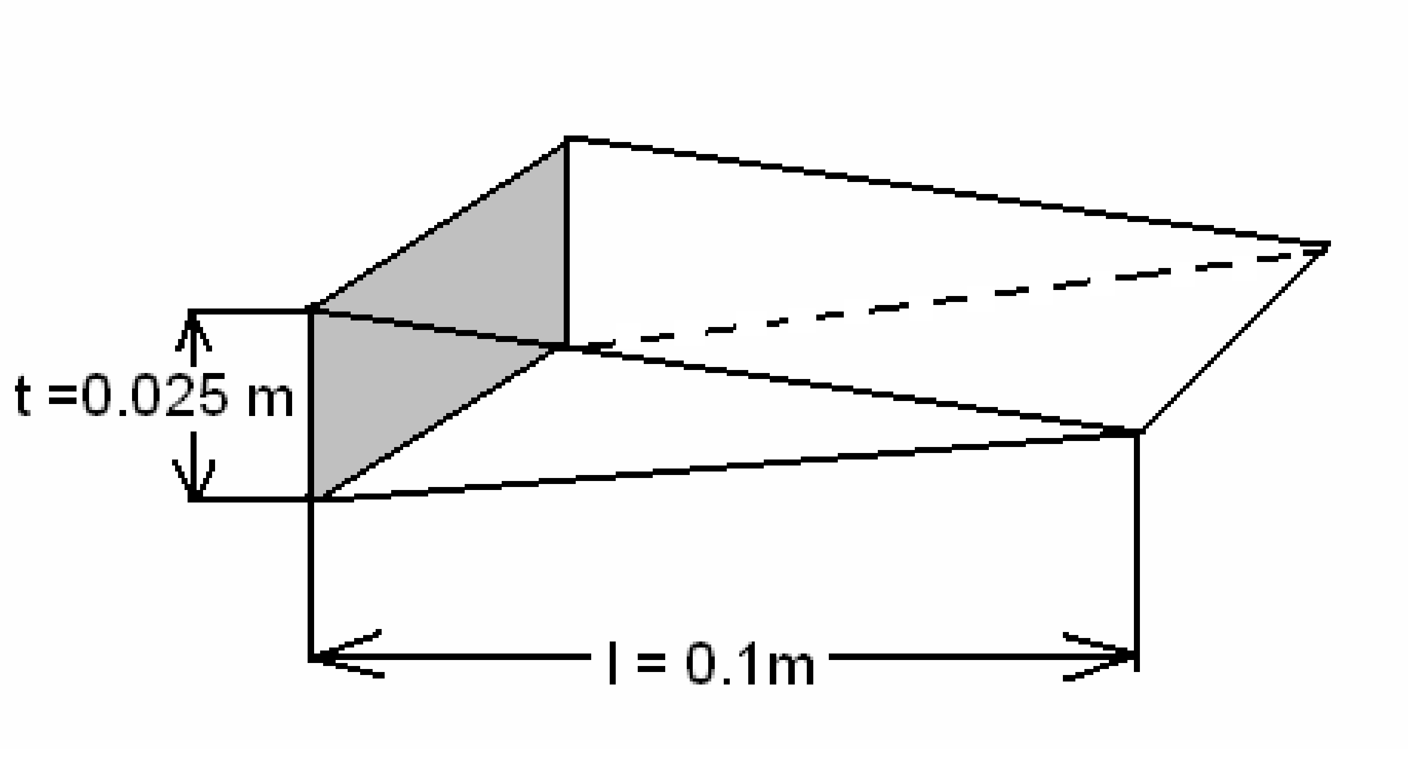
$$a_3 = 0.72411 \times 10^{-10}$$

$$\overline{U}(x) = x\,[2.5 \times 10^{-4} - 0.21846 \times 10^{-6}\,(600 - x)$$

$$- 0.72411 \times 10^{-10}\,(27 \times 10^{4} - x_2)$$

2) Heat transfer through Fin

Material - stainless steel	
Thermal conductivity	$K = 17.7$ W/mK
Film Coefficient	$h = 20.0$ $W/m^{2o}K$
Thickness at root	$t_o = 0.025$ m
Length	$L = 0.1$ m
Assume unit width	$b = 1.0$ m
Ambient temperature	$T_\infty = 40°$ C
Wall temperature	$T_o = 600°$ C
Tip temperature	$T_L = 40°$ C $= T_\infty$


t =0.025 m
l = 0.1m

Governing equation is

$$-\frac{d}{dx}\left[KA(x)\,\frac{dT(x)}{dx}\right] + hp\,[T(x) - T_\infty] = 0 \qquad \text{-----------} \; 1$$

Boundary Conditions

$$T(0) = T_0$$
$$T(L) = T_\infty$$

Let $\bar{T}(x) = T(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

Substituting the boundary conditions

$$a_0 = 600$$

$$a_1 = -5600 - 0.1\,a_2 - 0.01\,a_3 \qquad \text{-------------} \; 2$$

$$T(x) = 600 - 5600\,x + a_2\,x(x - 0.1) + a_3\,x\,(x^2 - 0.01)$$

The thickness at a point x-from the root,

$$t(x) = (1-x/L)$$

Substituting (2) in (1), the residue is given by

$$-\frac{d}{dx}\left[KA(x)\frac{dT(x)}{dx}\right] + hp\,[T(x) - T_\infty] \qquad \text{----------} \; 3$$

Collocation Method

Choosing points $X_1 = 0.03$ and $X_2 = 0.06$, and forcing the residue to be zero at these points.

i.e.
$$R(X_1) = 0$$
$$R(X_2) = 0$$

leads to a set of simultaneous equations

$$\begin{bmatrix} 0.88197 & 0.0991686 \\ 0.36246 & 0.0756936 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 8825.6 \\ 15730.4 \end{Bmatrix}$$

Solving for a_2 and a_3

$$a_2 = 28944.51$$

$$a_3 = -346418$$

substituting in (2) yields the approximation for the temperature distribution.

The closed form solution is given by

$$T(x) = 40 - 1502.3\sqrt{x} \qquad (6)$$

Comparison

x	$T_{cf}(x)$	$T_{app}(x)$
0.03	437.5	465.79
0.06	340.5	327.56

RITZ VARIATIONAL METHOD
(Weak Formulation)

Starting with the equation

$$\frac{d}{dx}\left[\alpha(\mathrm{x})\frac{\mathrm{du}}{\mathrm{dx}}\right] - \mathrm{f(x)} = 0 \text{ in } \Omega$$

The WR becomes

$$\int_{Xa}^{Xb} W(x)\left[\frac{\mathrm{d}}{\mathrm{dx}}\left\{\alpha(\mathrm{x})\frac{\mathrm{dU}}{\mathrm{dx}}\right\} - \mathrm{f(x)}\ \mathrm{dx} = 0\right.$$

W(x) -- weighting function

i.e., $\int \mathrm{R(x)\,w(x)\,dx}$

Observations:

- u is differentiated twice, while W(x) is remaining undifferentiated.
- So trial functions should be differentiable at least twice.
- But continuity of derivatives of higher order is very difficult.
- Hence preferable to reduce the order of derivatives of u as much as possible

This could be achieved by integration of the equation by parts.

$$\int_{Xa}^{Xb} W(x)\left[\frac{d}{dx}(\alpha(x)\frac{du}{dx})\right]dx = \left[W(x)\,(\alpha(x)\frac{du}{dx})\right]_{Xa}^{Xb}$$

$$-\int_{Xa}^{Xb} \alpha(x)\frac{du}{dx}\frac{dW}{dx}\,dx$$

The equation can be now recast at

$$\int_{Xa}^{Xb} \alpha(x)\frac{du}{dx}\frac{dW}{dx}\,dx = -\int_{Xa}^{Xb} f(x)\,W(x)\,dx$$

$$+\left[W(x)\left[\alpha(x)\frac{du}{dx}\right]\right]_{Xa}^{Xb}$$

i.e., B(u,W) = ℓ(W) B is the bilinear and ℓ is the linear

Recasting of the given differential equation in this form where the order of derivatives are traded between the trial function and the weighting function, thereby weakening the continuity requirement on the trial functions is called 'Weak Formulation'. The original equation is recast into its Weak Form.

The Ritz method we take, $W(x) = \delta U(x)$
Where u(x) is specified, as at the boundary, W(x) = 0.

APPLICATION OF VARIATIONAL FORMULATION

Illustrative Example for Variational Formulation

Consider the elastic deformation of a tapered - rod under its weight and also due to applied pull at the free-end, considered previously.

The governing equation is

$$\frac{d}{dx}\left[EA(X)\frac{du}{dx}\right] + \gamma A(x) = 0 \quad \text{in} \quad 0 < x < L$$

With B.Cs i) $u(0) = 0$

and

ii) At $x=l$ $\quad P = EA(x)\dfrac{du\,(x)}{dx}$

The WR formulation is

$$\int_0^L w(x) \left\{ \frac{d}{dx} \left[EA(x) \frac{du}{dx} \right] + \gamma A(x) \right\} dx = 0$$

where w(x) is the weighting Function and u(x) is the trial solution. Integrating by parts and r-arranging, we get

$$\int_0^L EA(x) \frac{du}{dx} \frac{dw}{dx} dx = \int_0^L \gamma A(x) w(x) dx - w(0) P(0) + w(L) P(L)$$

i.e. $\quad B(u, w) = \ell(w)$

since u(0) = 0 (specified), w = δ.u. at x = 0 vanishes

i.e. W(0) = 0 P(L) = P - specified

$$B(u, w) = \int_0^L EA(x) \frac{du}{dx} \frac{dw}{dx} \, dx$$

$$\ell(w) = \int_0^L \gamma A(x) \, w(x) \, dx + Pw(L)$$

Since the bilinear term B is symmetric [B (u, w) = B (w, u)] a quadratic functional I(u) exists and is given by I(u) = 1/2 B (u, u) - (u)

$$I(u) = \int_0^L \frac{1}{2} EA(x) \frac{du^2}{dx} dx \quad - \int_0^L \gamma A(x)\, u(x)\, dx - \rho\, \delta\, u(L)$$

strain-energy of deformation   External work by distributed load   External workdone by concentrated load

clearly I(u) gives the Total Potential of the elastic system, which is stationary

$$\delta\, I(u) = 0 = \int_0^L EA(x) \frac{du}{dx}\, \delta \frac{du}{dx}\, dx - \int_0^L \gamma\, A(c)\, \delta u(x)\, dx - \rho\, \delta\, u(L)$$

we know that $w(x) = \delta u(x)$ and threfore

$$\delta\left(\frac{du}{dx}\right) = \frac{d}{dx}(\delta u) = \frac{dw}{dx}$$

$\therefore$ We get

$$\int_0^L EA(x)\,\frac{du}{dx}\,\frac{dw}{dx}\,dx = \int_0^L \gamma\,A(x)\,w(x)\,dx - Pw(L)$$

$B(u, w) = \ell(w)$ - the weak form

Ritz Method of Solution

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Essential boundary condition is u(0) = 0

We get $a_o = 0$ and

$$u(x) = \sum_{j=1}^{3} a_j \phi_j(x) \qquad \text{where } \phi_j(x) = x^j$$

The weighting function is w(x) $= \phi_{i\,(x)}$ i = 1, 2, 3

substituting in the Weak-form of the governing equation.

This leads us to the equation

$$\sum_{j=1}^{3} a_j \int EA(x) \frac{d\phi_i}{dx} \frac{d\phi_i}{dx} dx = r_i \qquad \text{i = 1, 2, 3}$$

where $$r_i = \int_0^L \gamma A(x) \phi_i (x) dx + P \phi_i (L)$$

on evaluation of the integral within the brackets, this reduces to the set of algebraic equations.

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix}$$

Where $k_{ij} = \int_0^L EA(x) \dfrac{d\phi_i}{dx} \dfrac{d\phi_j}{dx}\, dx$

Solution of this matrix equation leads to determination of the constants a_1,a_2 and a_3 there by giving the approximate solution.

$$u(x) = \sum_{j=1}^{3} a_j x^j$$

For the given illustrative example of a tapered rod under its weight and also due to applied pull at the free-end

For the given illustrative example of a tapered rod under its weight and also due to applied pull at the free-end

when $i = 1, \quad j = 1$

when $i = 1, \quad j = 2 \ldots\ldots \quad k_{12}$

$i = 1, \quad j = 2 \ldots\ldots \quad k_{13}$

$$k_{11} = EA(x)\,\frac{d\phi_1}{dx}\,\frac{d\phi_1}{dx}\,dx$$

$$= E(80 - 0.2x).1.1dx = 1.5\ x10^4 \quad E$$

$$k_{12} = EA(x)\,\frac{d\phi_1}{dx}\,\frac{d\phi_2}{dx}\,dx$$

$$= E(80 - 0.2x).1.2x..dx = 8.6x10^8\ E$$

$$k_{13} = EA(x)\,\frac{d\phi_1}{dx}\,\frac{d\phi_3}{dx}\,dx$$

$$=E(80 - 0.2x).1.3x.2.\ dx =8.6\ x10^8\ E$$

Where k_{21} = ………. K_{22} = ………..

k_{23} = ………. K_{31} = ………..

$$r_1 = \gamma\, A(x)\, \phi_1\, dx = \gamma\, (80 - 0.2x)\ .x.\ dx = 1.3773 \times 10^5$$

$$r_2 = \gamma\, A(x)\, \phi_2\, dx = \gamma\, (80 - 0.2x)\ .x^2.\ dx = 1.3773 \times 10^7$$

$$r_3 = \gamma\, A(x)\, \phi_3\, dx = \gamma\, (80 - 0.2x)\ .x^3.\ dx = 1.3773 \times 10^9$$

$$p_1 = p.\, \phi_1\, (L) = p\, L = 3 \times 10^7$$

$$p_2 = p.\, \phi_2\, (L) = p\, L^2 = 9 \times 10^9$$

$$p_1 = p.\, \phi_3\, (L) = p\, L^3 = 27 \times 10^{11}$$

$$\begin{bmatrix} 1.5 \times 10^4 & 3.6 \times 10^6 & 9.45 \times 10^8 \\ 3.6 \times 10^6 & 1.2 \times 10^9 & 2.88 \times 10^{11} \\ 9.45 \times 10^8 & 3.88 \times 10^{11} & 1.322 \times 10^{14} \end{bmatrix} * \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.37 \times 10^5 + 3 \times 10^7 \\ 2.4 \times 10^7 + 9 \times 10^9 \\ 4.598 \times 10^9 + 27 \times 10^{11} \end{bmatrix}$$

On solving

$$a_1 = 6.6762 \times 10^{-5}$$

$$a_2 = -4.946 \times 10^{-8}$$

$$a_3 = 6.4736 \times 10^{-10}$$

$$U|_{x=2} = a_1(300) + a_2(300)^2 + a_3(300)^3 = 0.033056 \text{ cm}$$

But the Strength of Mat Method give $\delta = 0.0378$ cm

$$\frac{d^2}{dx^2}\left\{b(x)\,\frac{d^2u}{dx^2}\right\} + c(x)\,u = f(x) \quad 0 < x < L$$

Weak form of the above equation reduces to
B(u,w) = l(w)

$$\int_0^L [b(x)\frac{d^2u}{dx^2}\frac{d^2w}{dx^2} + c(x)\,uw\,]\,dx = \int_0^L f(x)\,w(x)dx + \frac{dw}{dx}b(x)\frac{d^2u}{dx^2}\,|$$

$$- W\frac{d}{dx}(b(x)\frac{d^2w}{dx^2}\,|$$

Denoting b(x) = $$\frac{d^2u}{dx^2} = M(x)$$

and

$$\frac{dM}{dx} = Q(x)$$

We have

$$\ell(w) = \int_0^L f(x)w(x)\,dx + \frac{dw}{dx} M(x) \,\Big| \; - w\,Q(x) \,\Big|$$

In the case of elastic beams

$b(x) = EI(x)$ - the flexural rigidity

$c(x) = K$ - stiffness of the elastic foundation for static problems.

$u(x)$ - Transverse displacement at any point

$M(x)$ - Bending moment

$Q(x)$ - Shear force

Looking at the boundary terms, the terms containing the weighting function viz. $\omega(x)$ and dw/ds represent the essential boundary conditions. i.e.

$w(x) = \delta\,(u(x))$ - Specification of transverse displacement, u

$\frac{dw}{dx}(x) = \delta\left(\frac{du}{dx}\right)$ - Specification of slope $\theta = \frac{du}{dx}$

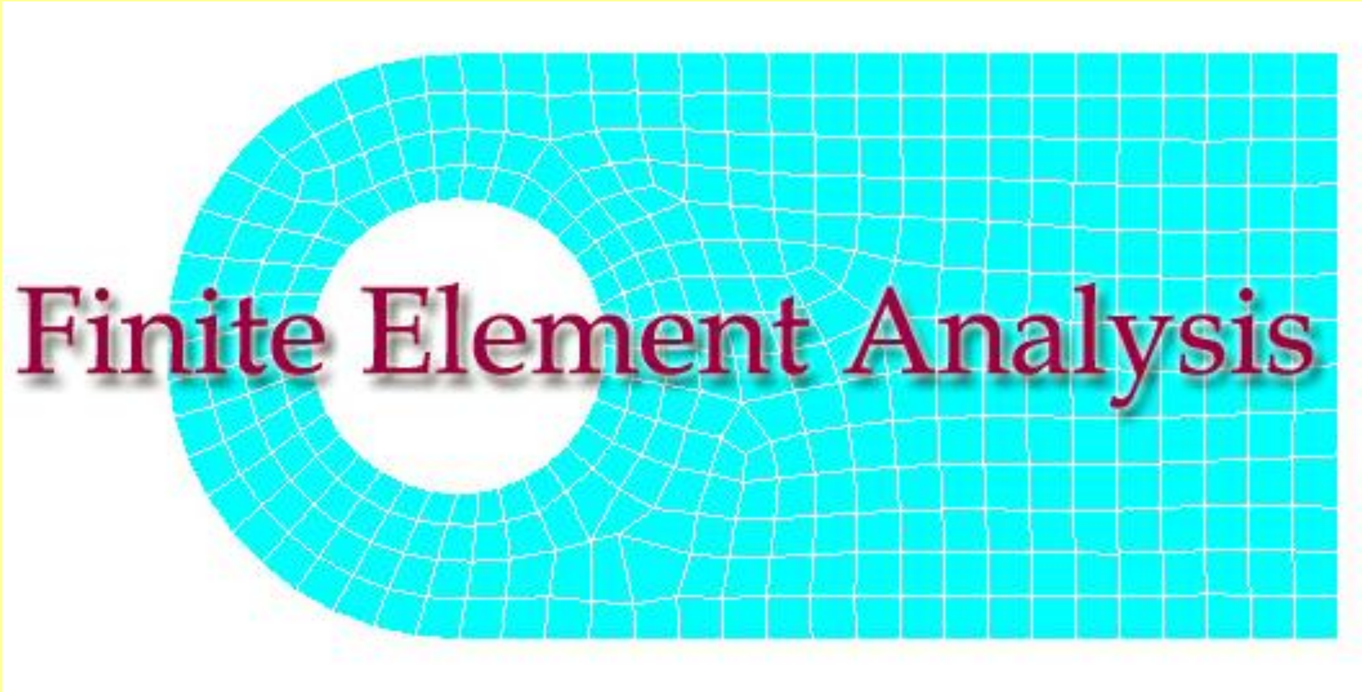
Since the bi-linear functional B(u, w) is symmetric, we have a quadratic functional that exists and is stationary. This functional is given by

$$I(u) = \frac{1}{2} B(u, u) - \ell(u)$$

$$= \frac{1}{2} [b(x) \left(\frac{du}{dx}\right)^2 + c(x) u^2] dx - f(x) u(x) dx - M(0) \theta(0)$$

$$+ M(L) \theta(L) - w(L) Q(L) + w(0) Q(0)$$

This is nothing but the Total Potential of the system which is a minimum at equilibrium configuration



LECTURE 3

RITZ VARIATIONAL METHOD
(Weak Formulation)
Steps:

i) Bring all the terms of the governing equation to one side of the equality

ii) Multiply with a weighting function w(x)

iii) Integrate by parts over the limits of the domain

iv) Separate linear and bilinear terms

v) Identify the boundary terms

RITZ VARIATIONAL METHOD (Weak Formulation)

Starting with the equation

$$\frac{d}{dx}\left[\alpha\,(\mathrm{x})\,\frac{\mathrm{du}}{\mathrm{dx}}\right] - \mathrm{f(x)} = 0 \text{ in } \Omega$$

The Weighted residue becomes

$$\int_{Xa}^{Xb} w(x)\left[\frac{\mathrm{d}}{\mathrm{dx}}\left\{\alpha\,(\mathrm{x})\,\frac{\mathrm{d}\bar{\mathrm{u}}}{\mathrm{dx}}\right\} - \mathrm{f(x)}\;\;\mathrm{dx}\;\;\right] = 0$$

w(x) -- weighting function

i.e., $\int\ \mathrm{R(x)\,w(x)\,dx}$

Observations:

- u is differentiated twice, while w(x) is remaining undifferentiated.
- So trial functions should be differentiable at least twice.
- But continuity of derivatives of higher order is very difficult.
- Hence it is preferable to reduce the order of derivatives of u as much as possible

We note that the first term is of the form

$$\int u\,dv = \quad uv\Big| - \int v\,du$$

Where u = $\mathrm{w(x)}$

And v = $\left[\ \alpha(\mathrm{x})\,\frac{\mathrm{du}}{\mathrm{dx}}\right]$

$$\int_{Xa}^{Xb} w(x)\left[\ \frac{\mathrm{d}}{\mathrm{dx}}(\alpha\,(\mathrm{x})\,\frac{\mathrm{du}}{\mathrm{dx}})\right]\mathrm{dx} = \left[\mathrm{w(x)}\ (\alpha\,(\mathrm{x})\,\frac{\mathrm{du}}{\mathrm{dx}})\right]_{Xa}^{Xb}$$

$$- \int_{\mathrm{Xa}}^{\mathrm{Xb}} \alpha(x)\,\frac{\mathrm{du}}{\mathrm{dx}}\,\frac{\mathrm{dw}}{\mathrm{dx}}\,\mathrm{dx}$$

The equation can be now recast as

$$\int_{Xa}^{Xb} \alpha(x)\,\frac{\mathrm{du}}{\mathrm{dx}}\,\frac{\mathrm{dw}}{\mathrm{dx}}\,\mathrm{dx} = \int_{\mathrm{Xa}}^{\mathrm{Xb}} f(x)\,\mathrm{w(x)}\,\mathrm{dx} + \left[\mathrm{w(x)}\left[\alpha(\mathrm{x})\,\frac{\mathrm{du}}{\mathrm{dx}}\right]\right]_{Xa}^{Xb}$$

Now $\displaystyle\int_{Xa}^{Xb} \alpha(x)\,\frac{\mathrm{du}}{\mathrm{dx}}\,\frac{\mathrm{dw}}{\mathrm{dx}}\,\mathrm{dx}$

is a linear function of both field variable and weighting function = B(u,w)

And $$-\int_{Xa}^{Xb} f(x)\, w(x)\, dx$$

is a function of weighting function alone

$$\left[w(x) \left[\alpha(x) \frac{du}{dx} \right] \right]_{Xa}^{Xb}$$

Represents the boundary term where

$$\left[\alpha(x) \frac{du}{dx} \right]$$

Is the flux or secondary variable

i.e., B(u,w) = ℓ(w)

B is the bilinear function and ℓ is the linear function

$$\int_{Xa}^{Xb} \alpha(x) \frac{\mathrm{du}}{\mathrm{dx}} \frac{\mathrm{dw}}{\mathrm{dx}} \mathrm{dx} = - \int_{\mathrm{Xa}}^{\mathrm{Xb}} f(x) \mathrm{w(x)}\, \mathrm{dx} + \left[\mathrm{w(x)} \left[\alpha(\mathrm{x}) \frac{\mathrm{du}}{\mathrm{dx}} \right] \right]_{Xa}^{Xb}$$

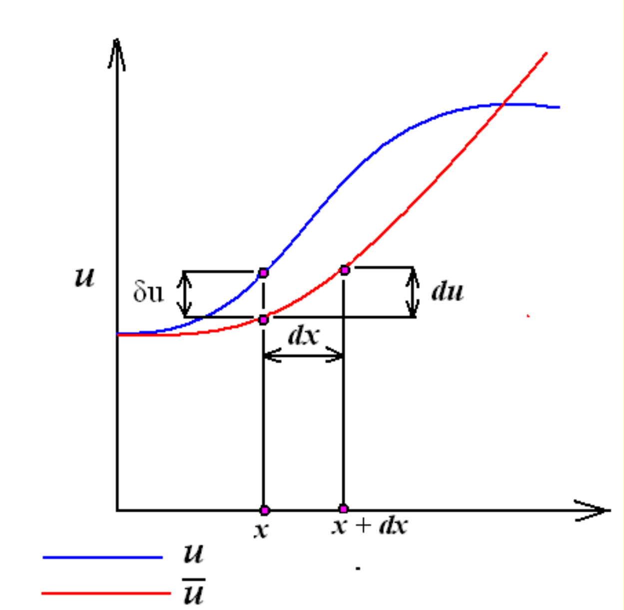
The above represents the weak form of the original Governing equation

Recasting of the given differential equation in this form where the order of derivatives are traded between the trial function and the weighting function, thereby weakening the continuity requirement on the trial functions is called ‘Weak Formulation’.

The original equation is recast into its Weak Form.

In the Ritz method we take, $w(x) = \delta u(\mathrm{x})$ which implies that where ever u(x) is specified, as at the boundary, w(x) = 0.

$w(x) = \delta u(\mathrm{x})$ Represents the variation of the field variable.



APPLICATION OF VARIATIONAL FORMULATION

Illustrative Example for Variational Formulation

Consider the elastic deformation of a tapered - rod under its weight and also due to applied pull at the free-end, considered previously.

The governing equation is

$$\frac{d}{dx}[EA(x)\ \frac{du}{dx}] + \gamma A(x) = 0 \qquad \text{in} \quad 0 < x < L$$

With B.Cs i) u(0) = 0

and

ii)At x=l $\quad [EA(x)\ \frac{du}{dx}] = P$

The WR formulation is

$$\int_0^L w(x) \left\{ \frac{d}{dx} \left[EA(x) \frac{du}{dx} \right] + \gamma A(x) \right\} dx = 0$$

where w(x) is the weighting function and u(x) is the trial solution. Integrating by parts and r-arranging, we get

$$\int_0^L EA(x) \frac{du}{dx} \frac{dw}{dx} dx = \int_0^L \gamma A(x) w(x) dx - w(0) P(0) + w(L) P(L)$$

i.e. $\quad B(u, w) = l(w)$

since u(0) = 0 (specified), w = δu at x = 0 vanishes

i.e. w(0) = 0 P(L) = P - specified

Hence P(0) w(0) term vanishes

$$B(u, w) = \int_0^L EA(x) \frac{du}{dx} \frac{dw}{dx} dx$$

$$\ell(w) = \int_0^L \gamma A(x) w(x) dx + Pw(L)$$

Since the bilinear term B is symmetric ie. [B(u,w) = B (w,u)] a quadratic functional I(u) exists and is given by I(u) = 1/2 B (u, u) - *l*(u)

$$I(u) = \int_0^L \frac{1}{2} EA(x) \frac{du}{dx}^2 dx \quad - \int_0^L \gamma A(x)\, u(x)\, dx - \rho\, \delta\, u(L)$$

strain-energy of deformation | External work by distributed load | External workdone by concentrated load

clearly I(u) gives the Total Potential of the elastic system, which is stationary

$$\delta\, I(u) = 0 = \int_0^L EA(x) \frac{du}{dx} \delta \frac{du}{dx} dx - \int_0^L \gamma\, A(c)\, \delta u(x)\, dx - \rho\, \delta\, u(L)$$

we know that $w(x) = \delta u(x)$ and threfore

$$\delta\left(\frac{du}{dx}\right) = \frac{d}{dx}(\delta u) = \frac{dw}{dx}$$

$\therefore$ We get

$$\int_0^L EA(x)\,\frac{du}{dx}\,\frac{dw}{dx}\,dx = \int_0^L \gamma\,A(x)\,w(x)\,dx - Pw(L)$$

$B(u, w) = \ell(w)$ - the weak form

Advantages of the weak form

- **Order of the differential equation becomes half of that in the original equation.**

- **Hence continuity requirements on the assumed solution is reduced.**

- **Lower order polynomial can be assumed for the approximate solution.**

- **The Natural Boundary condition becomes embedded in the weak form**

- **Hence the trial solution needs to satisfy only the essential boundary condition**

Ritz Method of Solution

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Essential boundary condition is u(0) = 0

We get $a_o = 0$ and

$$u(x) = \sum_{j=1}^{3} a_j \phi_j(x) \qquad \text{where } \phi_j(x) = x^j$$

The weighting function is w(x) $=\phi_i(x)$ i = 1, 2, 3

substituting in the Weak-form of the governing equation.

This leads us to the equation

$$\sum_{j=1}^{3} a_j \int EA(x) \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx = r_j \qquad i = 1, 2, 3$$

where $$r_j = \int_0^L \gamma A(x)\, \phi_j(x)\, dx + P\, \phi_j(L)$$

on evaluation of the integral within the brackets, this reduces to the set of algebraic equations.

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix}$$

Where $k_{ij} = \int_0^L EA(x) \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx$

Solution of this matrix equation leads to determination of the constants a_1,a_2 and a_3 there by giving the approximate solution.

$$u(x) = \sum_{j=1}^{3} a_j x^j$$

For the given illustrative example of a tapered rod under its weight and also due to applied pull at the free-end

when $i = 1, \; j = 1 \ldots.\; k_{11}$

when $i = 1, \; j = 2 \ldots.\; k_{12}$

$i = 1, \; j = 2 \ldots.\; k_{13}$

and so on

$$k_{11} = \int_0^{300} EA(x) \frac{d\phi_1}{dx} \frac{d\phi_1}{dx} dx$$

$$= E(80 - 0.2x).1.1dx = 1.5 \text{ x}10^4 \quad E$$

$$k_{12} = \int_0^{300} EA(x) \frac{d\phi_1}{dx} \frac{d\phi_2}{dx} dx$$

$$= E(80 - 0.2x).1.2x.dx = 3.6\text{x}10^6 \ E$$

$$k_{13} = \int_0^{300} EA(x) \frac{d\phi_1}{dx} \frac{d\phi_3}{dx} dx$$

$$=E(80 - 0.2x).1.3x^2 \ dx =8.6 \text{ x}10^8 \ E$$

Similarly

$k_{21} = 3.6x10^6$ E $\quad$ $k_{22} = 1.2$ x10^9 E

$k_{23} = 2.88$ x10^{11} E $\quad$ $k_{31} = 8.6$ x10^8 E

$k_{32} = 2.88$ x10^{11}E $\quad$ $k_{33} = 1.322$ x10^{14} E

$$r_1 = \int \gamma\, A(x)\, \phi_1\, dx = \int \gamma\, (80 - 0.2x)\ .x.\ dx = 1.3773 \times 10^5$$

$$r_2 = \int \gamma\, A(x)\, \phi_2\, dx = \int \gamma\, (80 - 0.2x)\ .x^2.\ dx = 24 \times 10^7$$

$$r_3 = \int \gamma\, A(x)\, \phi_3\, dx = \int \gamma\, (80 - 0.2x)\ .x^3.\ dx = 4.598 \times 10^9$$

$$p_1 = P.\,\phi_1\,(L) = P\,L = 3 \times 10^7$$

$$p_2 = P.\,\phi_2\,(L) = P\,L^2 = 9 \times 10^9$$

$$p_1 = P.\,\phi_3\,(L) = P.\,L^3 = 27 \times 10^{11}$$

$$\begin{bmatrix} 1.5 \times 10^4 & 3.6 \times 10^6 & 9.45 \times 10^8 \\ 3.6 \times 10^6 & 1.2 \times 10^9 & 2.88 \times 10^{11} \\ 9.45 \times 10^8 & 2.88 \times 10^{11} & 1.322 \times 10^{14} \end{bmatrix} * \begin{bmatrix} \mathrm{a}_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.37 \times 10^5 + 3 \times 10^7 \\ 2.4 \times 10^7 + 9 \times 10^9 \\ 4.598 \times 10^9 + 27 \times 10^{11} \end{bmatrix}$$

On solving

$$a_1 = 6.6762 \times 10^{-5}$$

$$a_2 = -4.946 \times 10^{-8}$$

$$a_3 = 6.4736 \times 10^{-10}$$

$$U\big|_{x=300} = a_1(300) + a_2(300)^2 + a_3(300)^3 = 0.033056 \text{ cm}$$

But the Strength of Material Method gives deflection at the tip as = 0.0378 cm

THE FINITE ELEMENT METHOD or NODAL APPROXIMATION METHOD:

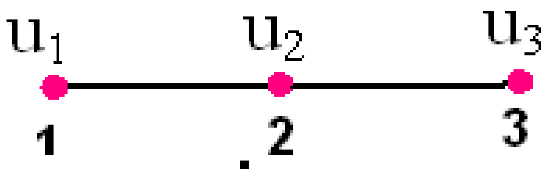
- The basic concept behind the Finite element method is "going from part to whole"
- Name **"FINITE ELEMENT"** coined by Clough
- Fitting of a number of piecewise continuous polynomials to approximate the variation of the field variable over the entire domain

STEPS INVOLVED IN THE FINITE ELEMENT METHOD:

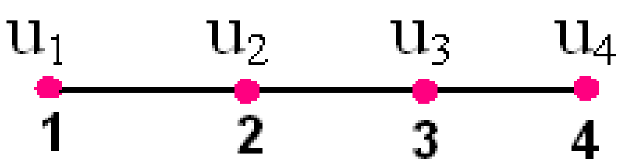
Discretisation of the structure: In this step the given structure is divided into subdivisions or elements. Depending upon the problem we may choose I D, II D or IIID elements.

I D elements

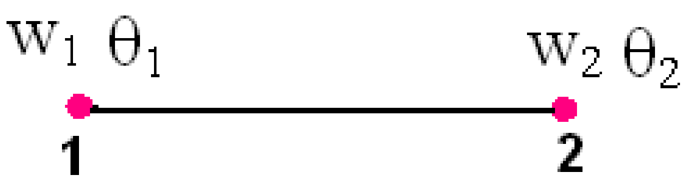
2 NODED LINEAR ELEMENT



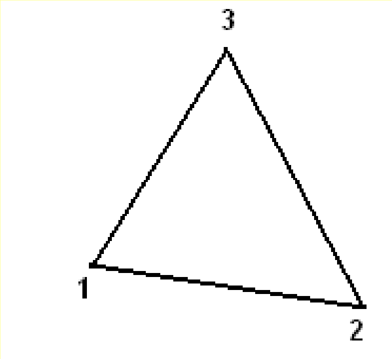
3 NODED QUADRATIC ELEMENT



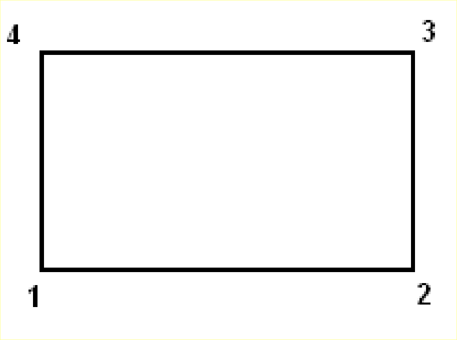
4 NODED CUBIC ELEMENT



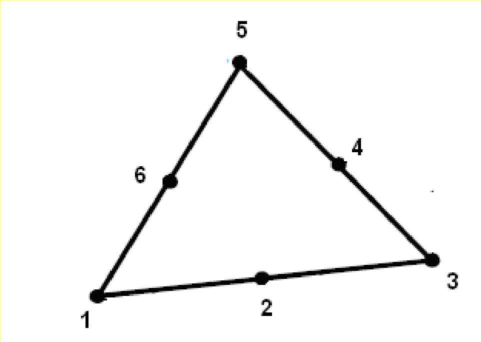
2 NODED BEAM ELEMENT



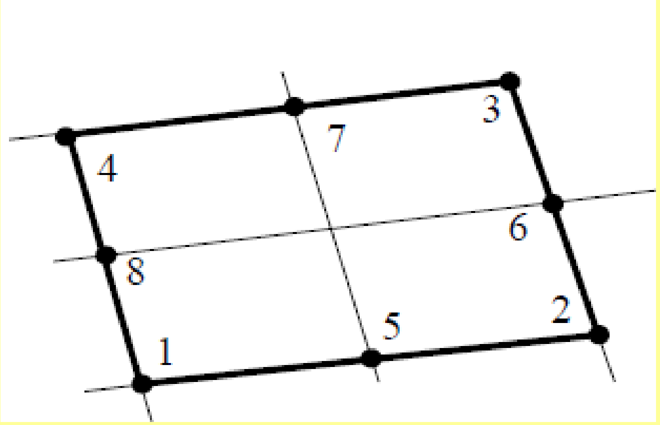
Constant strain triangular element



Bilinear Rectangular element

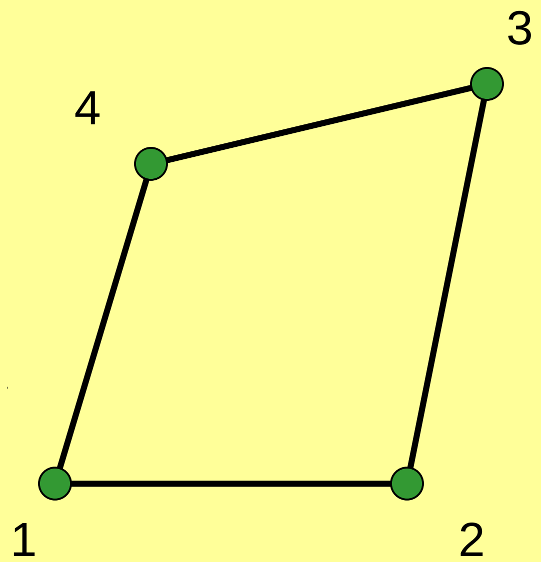


Linear strain triangular element



Eight noded quadratic quadrilateral elemen

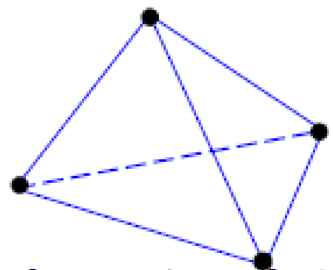
II D elements



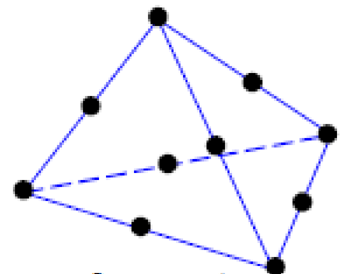
Linear Quadrilateral element

III D elements

Tetrahedron:

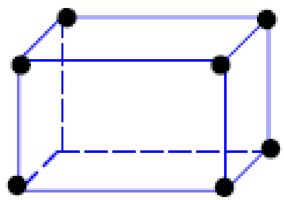


linear (4 nodes)

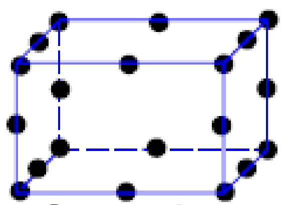


quadratic (10 nodes)

Hexahedron (brick):

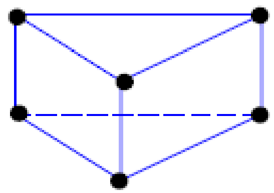


linear (8 nodes)

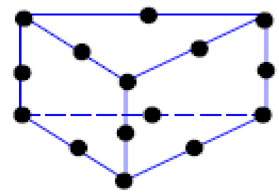


quadratic (20 nodes)

Penta:



linear (6 nodes)



quadratic (15 nodes)

Selection of suitable displacement model:

We make an assumption as to the variation of the unknown solutions over the element. In general, the field variable (example, temperature, displacement etc) is assumed to vary linearly or quadratically or cubically.

Displacement model associated with each element

For n = 1 (Linear model)

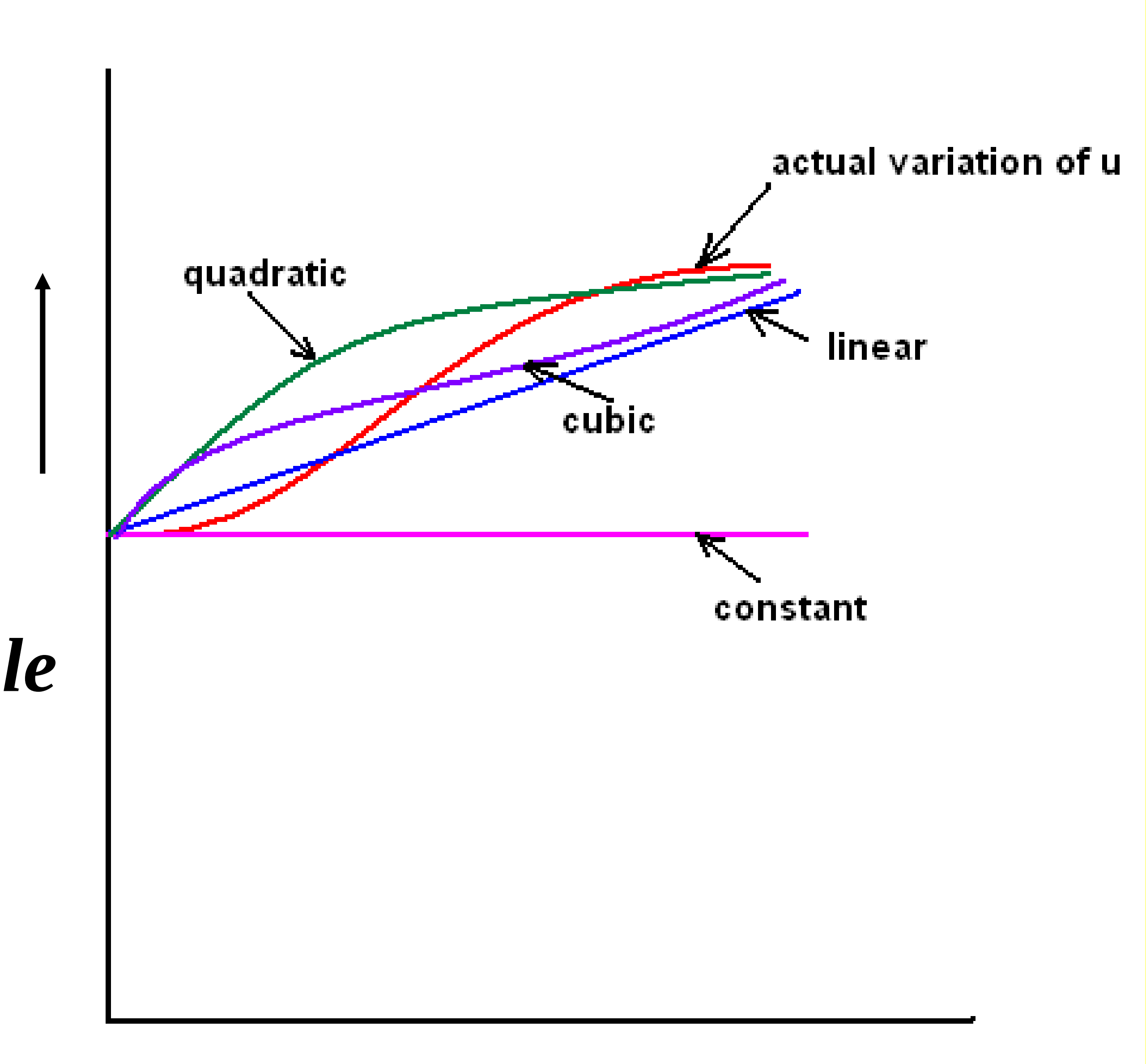
$$u(x) = a_0 + a_1 x$$

For n = 2 (quadratic model)

$$u(x) = a_0 + a_1 x + a_2 x^2$$

For n = 3 (cubic model)

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$



Field variable u

Length l

Derivation of elemental matrices and load vectors:

From the assumed displacement model, the elemental stiffness matrix $[K]^e$ and load vector $[P]^e$ of the element are to be derived using either equilibrium methods or a suitable variational principle.

Assembly of elemental equations to obtain overall stiffness matrix: the individual element stiffness matrices and load vectors are to be assembled in a suitable manner to get the overall stiffness equation which is expressed as

$$[K]\{u\} = \{P\}$$

where [K] is the assembled stiffness matrix

{u} is the vector of unknowns or nodal displacements

{P} is the vector of nodal forces for the complete structure

Imposition of boundary conditions: The Boundary conditions could now be incorporated to get the reduced equations.

Solutions for the unknown nodal displacements: The elemental matrices, on assembly, yield a set of equations, which could be expressed as a set of matrices, which could be solved using any iterative procedure or numerical method.

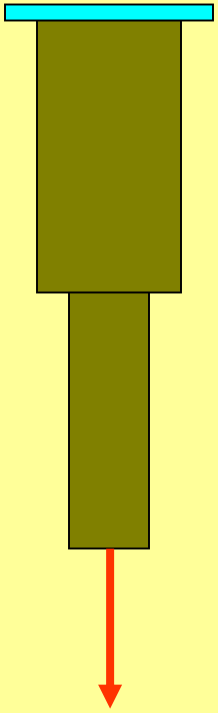
Computation of elemental strains and stresses: From the unknown displacements, the element strains and stresses can be computed by using the necessary equations of solid or structural mechanics.

$L_1 = 10\ cm$

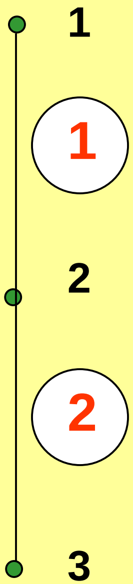
$L_2 = 10cm$

$E = 2x10^7 N/cm^2$

$A_1 = 2sq.cm$
$A_2 = 1sq.cm$



10 kN



EBC:

$U_1 = 0$

$P_l = 10kN$

$$[K]^1 = \frac{EA_1}{\ell_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K]^1 = \begin{bmatrix} 4 \text{ x } 10^5 & -4 \text{ x } 10^5 \\ -4 \text{ x } 10^5 & 4 \text{ x } 10^5 \end{bmatrix}$$

$$[K]^2 = \frac{EA_2}{\ell_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K]^2 = \begin{bmatrix} 2 \text{ x } 10^5 & -2 \text{ x } 10^5 \\ -2 \text{ x } 10^5 & 2 \text{ x } 10^5 \end{bmatrix}$$

The assembled stiffness matrix is given by

$$[K]^g = 10^5 \begin{pmatrix} 4 & -4 & 0 \\ -4 & 4+2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

The load vectors are

$$\{P\}^1 = \begin{Bmatrix} R \\ 0 \end{Bmatrix}$$

where R is the reaction at the fixed end

$$\{P\}^2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\{P\} = \begin{Bmatrix} R \\ 0 \\ 10 \end{Bmatrix}$$

The overall equilibrium equation is given by
[K] {u} = {P}

or

$$2 \times 10^5 \begin{pmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} R \\ 0 \\ 10 \end{Bmatrix}$$

$$2 \times 10^5 \begin{pmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} R \\ 0 \\ 10 \end{Bmatrix}$$

$$2 \times 10^5 \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$u_2 = 0.25 \times 10^{-4}$ cm

$u_3 = 0.75 \times 10^{-4}$ cm

Strain for element 1 = ϵ_1

$= \partial u / \partial x$ for element 1

$= (u_2 - u_1) / \ell_1$

$= 0.25 \times 10^{-5}$

Strain for element 2 = ϵ_2

$= \partial u / \partial x$ for element 2

$= u_3 - u_2 / \ell_2$

$= 0.50 \times 10^{-5}$

The stresses in the elements are given by

Stress in element 1 = $\sigma_1 = \epsilon_1 E_1$
$= (0.25 \times 10^{-5})\ (2 \times 10^{7})$
$= 5\ kN/cm^2$

Stress in element 2 = $\sigma_2 = \epsilon_2 E_2$
$= (0.50 \times 10^{-5})\ (2 \times 10^{7})$
$= 10\ kN/cm^2$

COMPUTATION OF REACTION AT FIXED END:

$2 \times 10^5 [2 * u_1 - 2 * u_2] = R$

Substituting for $u_1$and u2 we get

Reaction R= 10kN

NODAL APPROXIMATIONS

In general problems arise in engineering where we seek an approximation $\bar{u}$ (x, y, z) to some exact function u(x, y, z) to any desired level of accuracy, i.e.

$$u(x, y, z) = \bar{u}(x, y, z)$$

Many times the approximate function is obtained as a series expansion of some known function with undetermined coefficients. e.g.

$$\bar{u}(x) = \sum_{i=0}^{n} a_i x^i$$ (power series)

or $u(x) = \sum_{i=1}^{n} (a_i \cos ix + b_i \sin x)$ (Trigonometric series)

In these expansions $\mathbf{a_i}$ - s are called the **“generalised coordinates”**

$u(x_i) = u_i$ $\quad$ i = 1,2,.....r. Forcing the approximations to take on these specified values at the specified points, we have

$$u_i = \langle 1 \quad x_i \quad x_i^{\,2} \;\ldots\; x_i^{\,n-1} \rangle \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ . \\ . \\ . \\ a_n \end{Bmatrix}$$

$$(n \times 1)$$

$$i = 1, 2, \ldots\ldots r.$$

Taking r = n. We have

$$\{f_i\} \begin{Bmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_n \end{Bmatrix} = [\,P_n\,] \begin{Bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{Bmatrix}$$

(n x 1) (n x 1)

where the vector of a_i s and matrix $[P_n]$ are known

There, if $[P_n]$ is non-singular,

$$\begin{Bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{Bmatrix} = [P_n]^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_n \end{Bmatrix}$$

$$and\ f(x) = \underset{(1\ x\ n)}{<1\ \ x\ \ x^2\ \ldots.x^{n-1}>}\ \underset{(n\,x\,n)}{[P_n]^{-1}}\ \underset{(n\,x\,1)}{\{u\}}$$

$$= \underset{(1\,x\,n)}{<N_1\ N_2\ldots.N_n>} \underset{(n\ x\ 1)}{\begin{Bmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_n \end{Bmatrix}}$$

- The last equation expresses the approximation in terms of the function values at selected points, as compared to the expansion in terms of the generalised coordinates.
- These selected points are called the **"nodal points"** and {f} is called nodal-variable vector.
- The functions **$N_i(x)$ are called the shape functions.**
- Finally $u(x)=N_i(x)u_i$ is called the Nodal Approximation. N_i – s are also called as interpolation functions.

Derivation of Shape function for two noded element:

1) Let $u(x) = a_i + a_2 x$ in $0 < x < l$

$$= \langle 1 \quad x \rangle \begin{Bmatrix} a1 \\ a2 \end{Bmatrix}$$

$u(0) = u_1$ and $u(1) = u_2$

Therefore

$$a_1 = u_1$$

$$a_1 + a_2 l = u_2$$

In matrix form $\begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & \ell \end{bmatrix} * \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & \ell \end{bmatrix}^{-1} * \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -\frac{1}{\ell} & \frac{1}{\ell} \end{bmatrix} * \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\therefore u(x) = \langle\ \mathbf{1} \quad \mathbf{x}\ \rangle \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{-1/\ell} & \mathbf{1/\ell} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \langle\ \mathbf{(1- x/\ell)} \quad \mathbf{x/\ell}\ \rangle \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

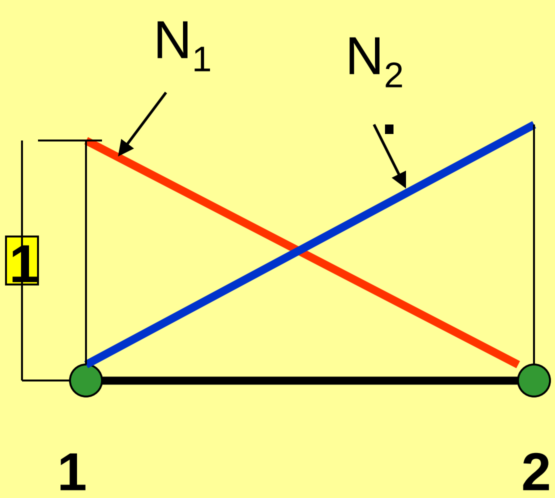
$$= \langle N1 \quad N2 \rangle \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = N_1 u_1 + N_2 u_2$$

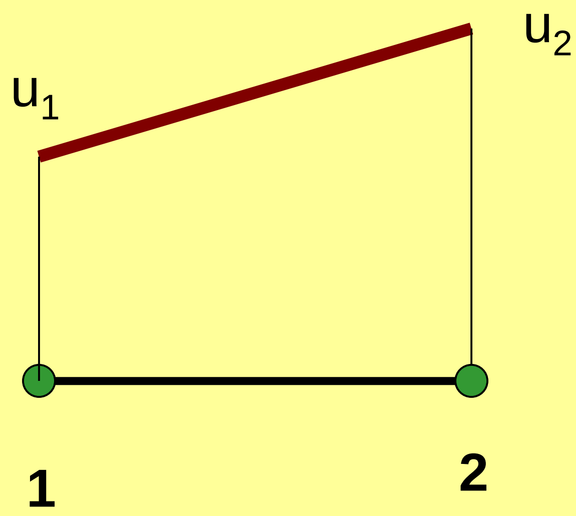
$$\boxed{\begin{array}{lll} N_1(x) = 1 - x/\ell & & N_1(0) = 1. \quad N_1(\ell) = 0 \\ N_1(x) = x/\ell & & N_2(0) = 0. \quad N_1(\ell) = 1 \\ & & N_1 + N_2 = 1 \end{array}}$$

It can be verified that

$$N_i(x_j) = 0 \qquad i \neq j$$
$$= 1 \qquad i = j$$
$$= \delta_{ij}$$

(Kronecker Delta Function)





To provide for the possibility of a constant or uniform field when f is constant at all points in the domain

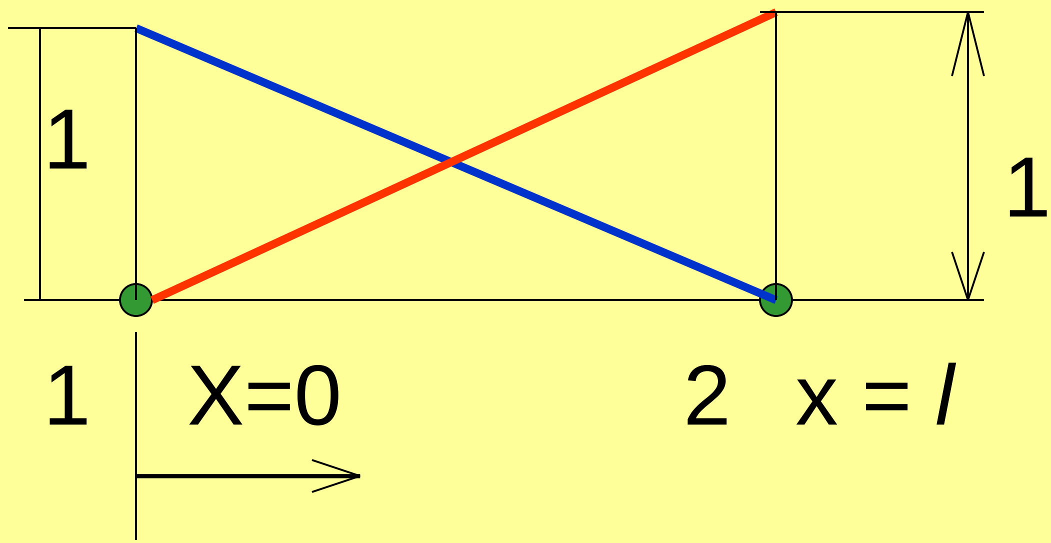
We have

$$\therefore \quad f(x) = C = \sum_{j=1}^{n} N_i (x) f_i = C \sum_{j=1}^{n} N_i (x)$$

$$f_1 = f_2 = \ldots\ldots\ldots = f_n = c$$

$$\therefore \quad \sum_{j=1}^{n} N_i (x) = 1$$

The above properties are very **important properties of shape functions.**



X

=

l

2) Let $u(x) = a_1 + a_2 x + a_3 x^2$

Shape functions for quadratic elements

$$= \langle 1 \quad x \quad x^2 \rangle \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Taking $x_1 = 0$, $x_2 = l/2$, $x_3 = l$ We have

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \ell/2 & \ell^2/4 \\ 1 & \ell & \ell^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/\ell & 4/\ell & -1/\ell \\ 2/\ell^2 & -4/\ell^2 & 2/\ell^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u(x) = \langle N_1 \quad N_2 \quad N_3 \rangle \begin{Bmatrix} \mathbf{u_1} \\ \mathbf{u_2} \\ \mathbf{u_3} \end{Bmatrix}$$

$$\mathbf{N_1 (x) \quad = 1 - 3x/\ell \quad + 2x^2/\ell^2}$$

$$\mathbf{N_2 (x) \quad = 4x/\ell \quad - 4x^2/\ell^2}$$

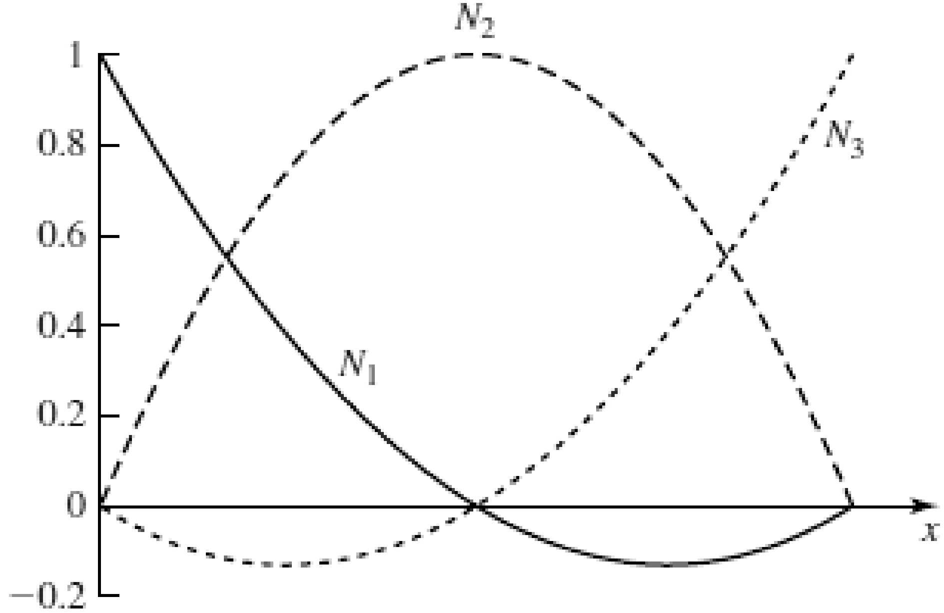
$$\mathbf{N_3 (x) \quad = -3/\ell \quad + 2x^2/\ell^2}$$

$$\mathbf{N_1 (0) = 1 \qquad N_1 (\ell/2) = 0 \qquad N_1 (\ell) = 0}$$

$$\mathbf{N_1 (0) = 1 \qquad N_1 (\ell/2) = 0 \qquad N_1 (\ell) = 0}$$

$$\mathbf{N_1 (0) = 1 \qquad N_1 (\ell/2) = 0 \qquad N_1 (\ell) = 0}$$

$$\mathbf{N_1 + N_2 + N_3 = 1}$$



Spatial variation of interpolation
functions for a three-node line element.

Finite Element Formulation

- In FEA, we use the nodal approximation to specify the unknown function in terms of its values at selected '***nodal points***', through a ***Nodal Approximation***

$$u(x) = \sum_{j=1}^{n} N_j(x) u_j \quad where$$

$$N_j - s \;\; are \;\; the \;\; "Interpolating" \;\; or \;\; "shape" \; functions$$

$$u_j - s \;\; are \;\; the \;\; values \;\; of \;\; 'u' \;\; t \;\; these \;\; nodal \;\; po\text{int}\,s$$

$$It \;\; is \;\; seen \;\; that \;\; the \;\; shape \;\; functions \;\; automatically$$

$$satisfy \;\; the \;\; specified \;\; essential \;\; boundary \;\; conditions$$

$$The \;\; weighting \;\; functions \;\; are \;\; chosen \;\; from \;\; the \;\; shape$$

$$functions; \quad \psi(x) = N_i(x) \qquad i = 1, 2, \ldots n$$

The governing equation is

$$\frac{d}{dx}[EA(x)\ \frac{du}{dx}] + \gamma A(x) = 0 \qquad \text{in} \qquad 0 < x < L$$

With B.Cs i) u(0) = 0

and

ii)At x=l $\quad [EA(x)\ \frac{du}{dx}] = P$

Weak form is given by

$$\int_0^L EA(x)\frac{du}{dx}\frac{dw}{dx}\,dx = \int_0^L \gamma\,A(x)\,w(x)\,dx + P(L)w(L) - P(0)\,w(0)$$

Substituting in the weak form

u(x) = $N_1u_1 + N_2u_2$

And w(x) as N_1 first and then N_2 we get a system of two equations in two unknowns namely u_1 and u_2

$$[K^e]\quad [u^e] \quad = \quad | \, r^e |$$

$$K_{ij}^e \quad = \int_0^{h_e} EA(x)\ \ (x)\, \frac{dN_i}{dx}\, \frac{dN_j}{dx}\, dx$$

$$r_j^e \quad = \int_0^{h_e} \gamma A(x) N_j \, dx + P_j$$

$$K_{11}^{e} = \int_{0}^{l} EA(x) \frac{dN_1}{dx}\frac{dN_1}{dx}\,dx$$

$$= \int E\,A\,(-1/l)(-1/l)dx$$

$$= EA/l^2 \int dx$$

$$= EA/l$$

$$K_{12}^{e} = \int_0^l EA(x) \frac{dN_1}{dx}\frac{dN_2}{dx} dx$$

$$= \int E\,A\,(-1/l)(1/l)dx$$

$$= -EA/l^2 \int dx = -EA/l$$

$$\begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \frac{\gamma l}{6} \begin{Bmatrix} 2A_1 + A_2 \\ 2A_2 + A_1 \end{Bmatrix}$$

$$= \int E\,A\,(-1/l)(1/l)dx$$

$$= -EA/l^2 \int dx = -EA/l$$

$$K_{22}^{e} = \int_{0}^{l} EA(x) \frac{dN_2}{dx} \frac{dN_2}{dx} dx$$

$$= \int E\,A\,(1/l)(1/l)dx$$

$$= EA/l^2 \int dx$$

$$= EA/l$$

Stiffness matrix for 2 noded element

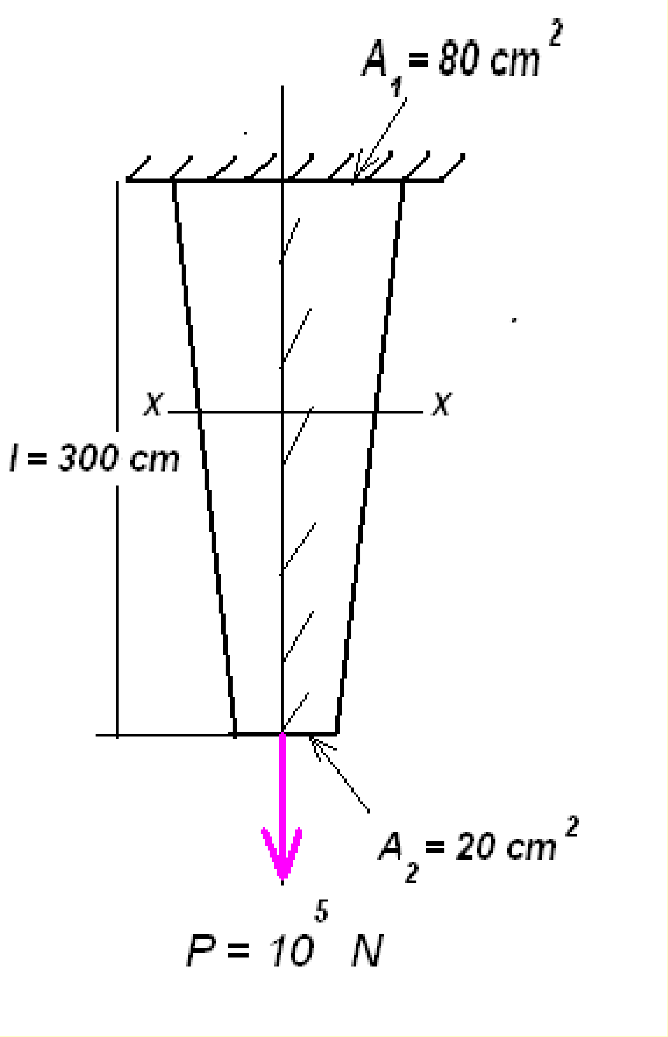
$$K = \frac{EA}{l} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$r_j^e = \int_0^{h_e} \gamma A N_j \, dx + P_j$$

$$r_1^e = \int_0^{h_e} \gamma A N_1 \, dx = \gamma\, Al/2$$

$$r_2^e = \int_0^{h_e} \gamma A N_2 \, dx = \gamma\, Al/2$$

$$\{r\} = \gamma\, Al/2 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$



$$A(x) = A_1 – (A_1 - A_2)\ x/l$$

$$\text{ie.}A(x) = 80 – (80-20)x/300$$

$$= (80 – 0.2x)$$

γ = 0.075 N/cm3

E = 2 x 107 N/cm2

If for the entire domain, there are only two nodal points, they also happen to be the boundary points x = 0 and x = L n = 2 and ij = 1, 2. The above equation reduces to

$$[K] \begin{Bmatrix} \mathbf{u_1} \\ \mathbf{u_2} \end{Bmatrix} = \begin{Bmatrix} \mathbf{r_1} \\ \mathbf{r_2} \end{Bmatrix}$$

2 x 2 2 x 1 2 x 1

Example

Consider the tapered rod problem

γ = 0.075 N/cm3 L =300cm

$E = 2 \times 10^7$ N/cm^2

$u_1 = 0$

$P_1 = R$

$P_2 = P$

$N_1(x) = 1 - x / L$ $N_2(x) = x / L$

$$\frac{dN_1}{dx} = -\frac{1}{L} \qquad \frac{dN_2}{dx} = -\frac{1}{L}$$

A(x) = 80 – 0.2x

$$K_{11} = \int_0^{300} E(80 - 0.2x)\left(-\frac{1}{L}\right)^2 dx = \frac{E}{L^2}\,(80\text{-}.02x)\ dx$$

$$= \frac{E}{L^2}\left[80L - \frac{0.2L^2}{2}\right]$$

$$= \frac{E}{L}(80 - 0.1L) = \frac{50E}{300} = \frac{E}{6}$$

$$K_{12} = K_{21} = -\frac{E}{6}$$

$$K_{22} = \qquad = \frac{E}{6}$$

$$[K] = \frac{E}{6} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$r_1 = \int_0^L \rho(80 - 0.2x)\ (1 - x/L)\ dx + R$$

$$= 675 + R$$

$$r_2 = \int_0^{300} \rho(80 - 0.2x)\ (x/L)\ dx + P$$

$$= 450 + 10^5$$

Apply the Boundary Condition $u_1 = 0$, this reduces to

$$k_{22}u_2 = r_2$$

$$u_2 = \frac{r_2}{K_{22}} = 0.03 \text{ cm}$$

This is the value of a uniform rod with average area under the pull. This compares with the Ritz method discussed earlier with a cubic polynomial which worked out to

$$u_2 = 0.033056 \text{ cm}$$

$$a_o + a_1 x = u(x)$$

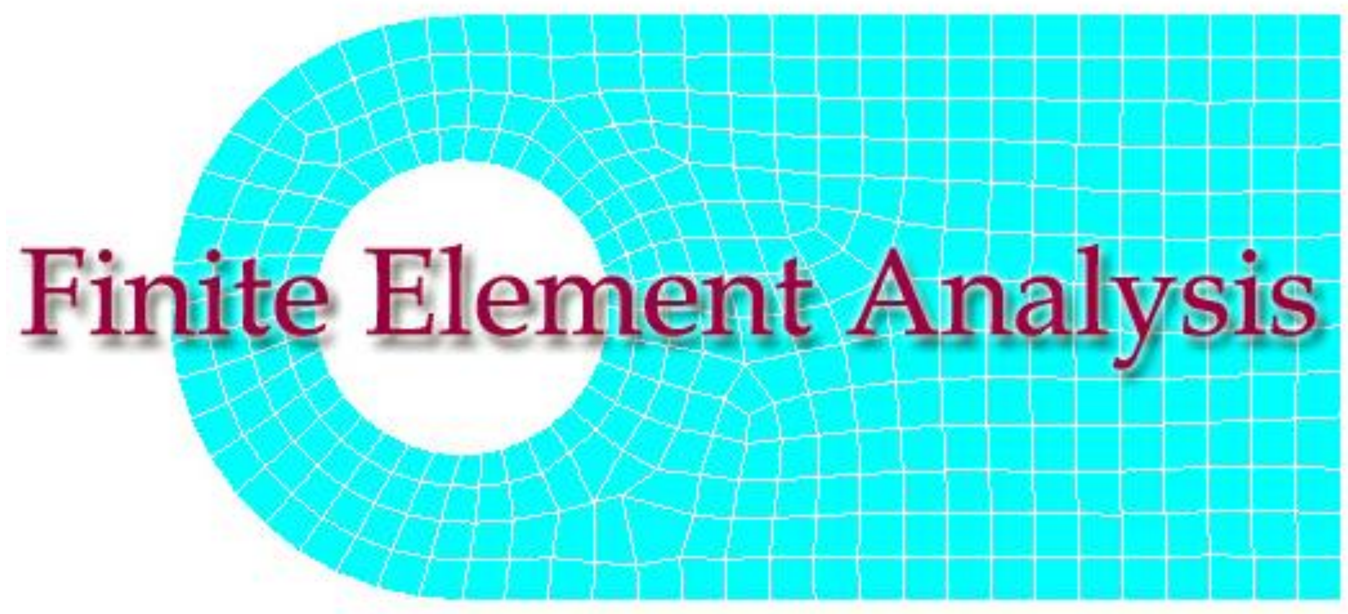
Linear displacement model

$$a_o + a_1 x + a_2 x^2 = u(x)$$

quadratic displacement model

$$a_o + a_1 x + a_2 x^2 + a_3 x^3 = u(x)$$

cubic displacement model



LECTURE 4

THE FINITE ELEMENT METHOD or NODAL APPROXIMATION METHOD:

- The basic concept behind the Finite element method is "going from part to whole"
- Name **"FINITE ELEMENT"** coined by Clough
- Fitting of a number of piecewise continuous polynomials to approximate the variation of the field variable over the entire domain

STEPS INVOLVED IN THE FINITE ELEMENT METHOD:

- ➢Discretisation of the structure
- ➢Selection of suitable displacement model
- ➢Derivation of elemental matrices and load vectors
- ➢Assembly of elemental equations to obtain overall stiffness matrix

STEPS INVOLVED IN THE FINITE ELEMENT METHOD:…contd

- Imposition of boundary conditions

- Solutions for the unknown nodal displacements

- Computation of elemental strains and stresses

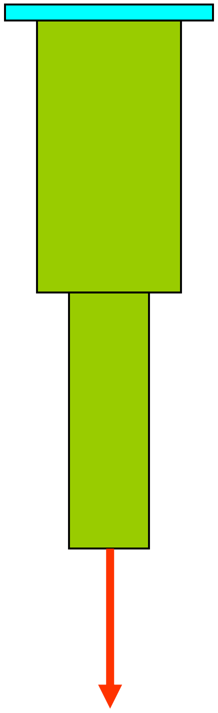
L_1 = 10 cm

L_2 = 10cm

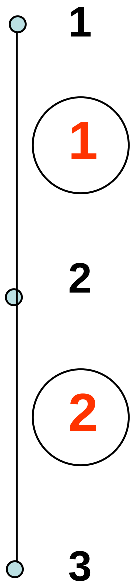
$E = 2x10^7 N/cm^2$

A_1 = 2sq.cm

A_2 = 1sq.cm



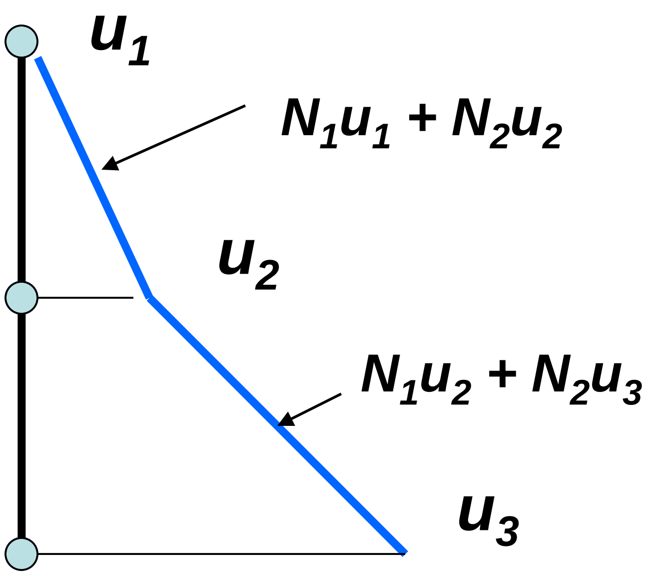
10 kN



BC:

$U_1 = 0$

P_l = 10kN


u_1
$N_1u_1 + N_2u_2$
u_2
$N_1u_2 + N_2u_3$
u_3

$$u(x) = a_1 + a_2 x$$

$$u(x) = N_1 u_1 + N_2 u_2$$

Here N_i s are called Shape functions or Interpolation functions
Shape functions are used to interpolate the field variable over the element in terms of nodal values of the field variable

$$N_1(x) = 1 - x/\ell \qquad N_1(0) = 1. \quad N_1(\ell) = 0$$

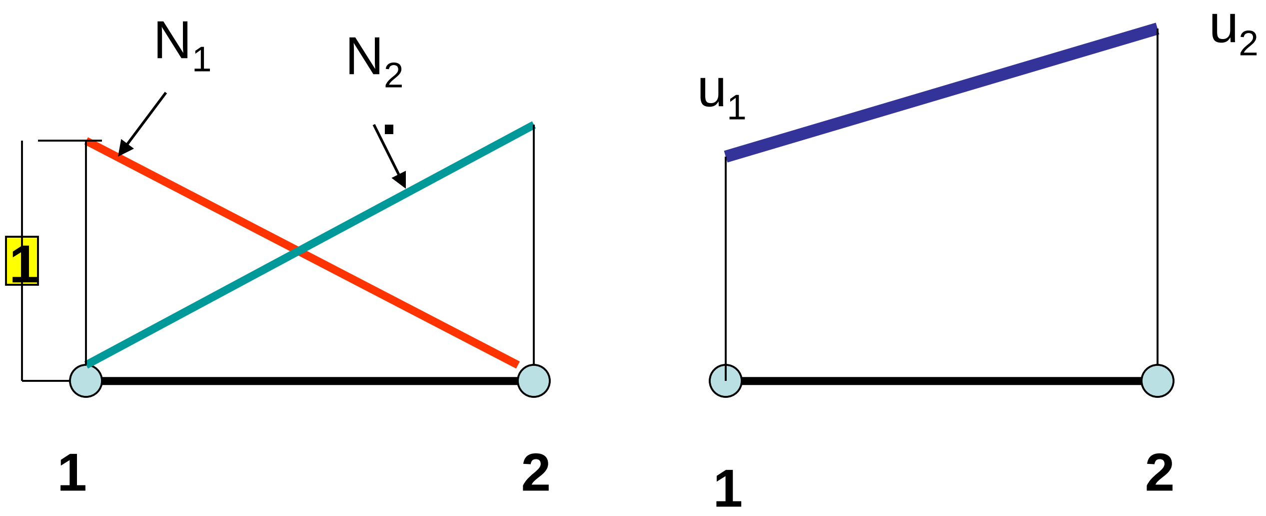
$$N_1(x) = x/\ell \qquad N_2(0) = 0. \quad N_1(\ell) = 1$$

$$N_1 + N_2 = 1$$

It can be verified that

$$N_i(x_j) \quad = \quad 0 \qquad i \neq j$$
$$= \quad 1 \qquad i = j$$
$$= \quad \delta_{ij}$$

(Kronecker Delta Function)



To provide for the possibility of a constant or uniform field when u is constant at all points in the domain

We have

$$\therefore \quad u(x) = c = \sum_{j=1}^{n} N_i\,(x)\,u_i = c\sum_{j=1}^{n} N_i\,(x)$$

$$u_1 = u_2 = \ldots\ldots. = u_n = c$$

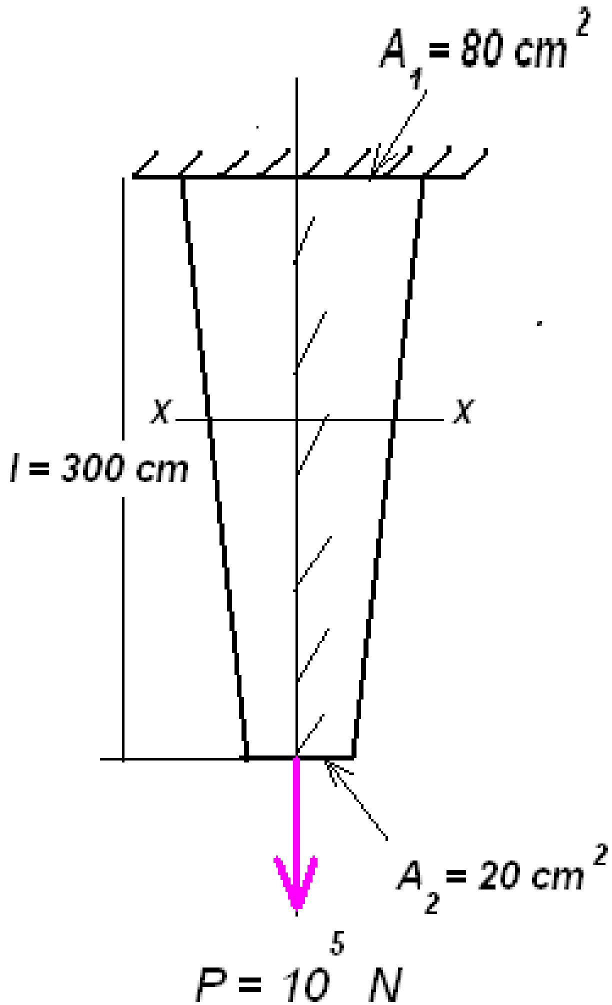
$$\therefore \quad N_1 c + N_2 c = c$$

$$or \sum_{j=1}^{n} N_i\,(x) = 1$$

The above properties are very important properties of shape functions.

- In FEA, we use the nodal approximation to specify the unknown function in terms of its values at selected ‘***nodal points***’, through a ***Nodal Approximation***

Now let us consider the numerical example of the tapered beam whose area of cross section varies uniformly from A_1 to A_2 at the free end and subjected to its own self weight and a point load at the end.



Example

$$A(x) = A_1 - (A_1 - A_2)\, x/l$$

$$\text{ie.}A(x) = 80 - (80-20)x/300$$

$$= (80 - 0.2x)$$

Specific weight $\gamma = 0.075\ N/cm^3$

Young's Modulus $E = 2 \times 10^7\ N/cm^2$

The governing equation is

$$\frac{d}{dx}[EA(x)\ \frac{du}{dx}] + \gamma A(x) = 0 \qquad \text{in} \quad 0 < x < L$$

With B.Cs i) $u(0) = 0$

and

ii)At x=l $\quad [EA(x)\ \frac{du}{dx}] = P$

Weak form is given by

$$\int_0^l EA(x)\,\frac{du}{dx}\,\frac{dw}{dx}\,dx = \int_0^l \gamma\,A(x)\,w(x)\,dx + P(l)w(l) - P(0)\,w(0)$$

Substituting in the weak form

$\boldsymbol{u(x) = N_1 u_1 + N_2 u_2}$

And w(x) as N_1 first and then N_2 we get a system of two equations in two unknowns namely u_1 and u_2

$$\int_0^l EA(x)\,\frac{d(N_1 u_1 + N_2 u_2)}{dx}\,\frac{dN_1}{dx}\,dx =$$

$$\int_0^l \gamma\, A(x)\; N_1\, dx + P(l)w(l) - P(0)\, w(0)$$

----1

$$\int_0^l EA(x)\,\frac{d(N_1 u_1 + N_2 u_2)}{dx}\,\frac{dN_2}{dx}\,dx =$$

$$\int_0^l \gamma\, A(x)\; N_2\, dx + P(l)w(l) - P(0)\, w(0)$$

----2

$$\int_0^l EA(x)\frac{\mathrm{d}(\mathrm{N}_1)}{\mathrm{dx}}\frac{\mathrm{dN}_1}{\mathrm{dx}}\,\mathrm{dx}\,u_1 + \int_0^l EA(x)\frac{\mathrm{d}(N_2)}{\mathrm{dx}}\frac{\mathrm{dN}_1}{\mathrm{dx}}\,\mathrm{dx}\;u_2 =$$

$$\int_0^l \gamma\,\mathrm{A(x)}\;\mathrm{N}_1\;\mathrm{dx} + \mathrm{P}(l)\mathrm{w}(l) - \mathrm{P(0)\;w(0)}$$

$$\int_0^l EA(x)\frac{\mathrm{d}(\mathrm{N}_1)}{\mathrm{dx}}\frac{\mathrm{dN}_2}{\mathrm{dx}}\,\mathrm{dx}\,u_1 + \int_0^l EA(x)\frac{\mathrm{d}(N_2)}{\mathrm{dx}}\frac{\mathrm{dN}_2}{\mathrm{dx}}\,\mathrm{dx}\;u_2 =$$

$$\int_0^l \gamma\,\mathrm{A(x)}\;\mathrm{N}_2\;\mathrm{dx} + \mathrm{P}(l)\mathrm{w}(l) - \mathrm{P(0)\;w(0)}$$

$$\underbrace{\int_0^l EA(x)\frac{d(N_1)}{dx}\frac{dN_1}{dx}\,dx}_{K_{11}}\,u_1 + \underbrace{\int_0^l EA(x)\frac{d(N_2)}{dx}\frac{dN_1}{dx}\,dx}_{K_{12}}\,u_2 =$$

$$\int_0^l \gamma A(x)\,N_1\,dx + P(l)w(l) - P(0)\,w(0)$$

$$\underbrace{\int_0^l EA(x)\frac{d(N_2)}{dx}\frac{dN_1}{dx}\,dx}_{K_{21}}\,u_1 + \underbrace{\int_0^l EA(x)\frac{d(N_2)}{dx}\frac{dN_2}{dx}\,dx}_{K_{22}}\,u_2 =$$

$$\int_0^l \gamma A(x)\,N_2\,dx + P(l)w(l) - P(0)\,w(0)$$

These 2 equations can be written in matrix form as

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \mathrm{u}_1 \\ \mathrm{u}_2 \end{Bmatrix} = \begin{Bmatrix} \mathrm{r}_1 \\ \mathrm{r}_2 \end{Bmatrix}$$

$$[K^e] \quad \{\mathrm{u}^e\} = \{\mathrm{r}^e\}$$

Where

$$K^e_{ij} = \int_0^l \mathrm{EA(x)} \ \frac{\mathrm{dN_i}}{\mathrm{dx}} \frac{\mathrm{dN_j}}{\mathrm{dx}} \, \mathrm{dx}$$

$$r^e_j = \int_0^l \gamma \mathrm{A}(x) \mathrm{N_j} \, \mathrm{dx}$$

We know that the shape functions for a two noded element are given by

$$N_1 = 1 - \frac{x}{l} \qquad N_2 = \frac{x}{l}$$

$$\frac{dN_1}{dx} = -\frac{1}{l} \qquad \frac{dN_2}{dx} = \frac{1}{l}$$

$$K_{11} = \int_0^l EA(x) \frac{dN_1}{dx} \frac{dN_1}{dx} dx$$

$$= \int_0^l E \left\{ A_1 - \frac{A_1 - A_2}{l} x \right\} \left(-\frac{1}{l} \right)^2 dx$$

$$= \frac{E}{l} \left(\frac{A_1}{2} + \frac{A_2}{2} \right) = \frac{E(A_1 + A_2)}{2l}$$

$$K_{12} = \int_0^l EA(x)\,\frac{dN_1}{dx}\frac{dN_2}{dx}\,dx$$

$$= \int_0^l E\left\{A_1 - \frac{A_1 - A_2}{l}\right\}x\left(-\frac{1}{l}\right)\left(\frac{1}{l}\right)dx$$

$$= -\frac{E}{l}\left(\frac{A_1}{2} + \frac{A_2}{2}\right) = -\frac{E(A_1 + A_2)}{2l}$$

$$K_{12} = K_{21}$$

$$K_{22} = \int_0^l EA(x) \frac{dN_2}{dx}\frac{dN_2}{dx} dx$$

$$= \int_0^l E \left\{ A_1 - \frac{A_1 - A_2}{l} \right\} \left(\frac{1}{l} \right)^2 dx$$

$$= \frac{E}{l} \left(\frac{A_1}{2} + \frac{A_2}{2} \right) = \frac{E(A_1 + A_2)}{2l}$$

Therefore the element stiffness matrix will be

$$[K^e] = \frac{E}{l} \frac{A_1 + A_2}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Similarly the element nodal load vector will be

$$r_1 = \int_0^l \gamma A(x)\, \mathrm{N}_1 \; \mathrm{dx}$$

$$= \int_0^l \left[\gamma\left\{A_1 - \frac{(\mathrm{A}_1 - \mathrm{A}_2)}{\mathrm{l}}\right\}(1 - \frac{\mathrm{x}}{\mathrm{l}})\right] dx$$

$$= \left\{\frac{\mathrm{A}_1}{3} l + \frac{\mathrm{A}_2}{6} l\right\}$$

$$r_2 = \int_0^l \gamma A(x)\, \mathrm{N}_2 \; \mathrm{dx}$$

$$= \int_0^l \left[\gamma\left\{A_1 - \frac{(\mathrm{A}_1 - \mathrm{A}_2)}{\mathrm{l}}\right\}(\frac{\mathrm{x}}{\mathrm{l}})\right] dx$$

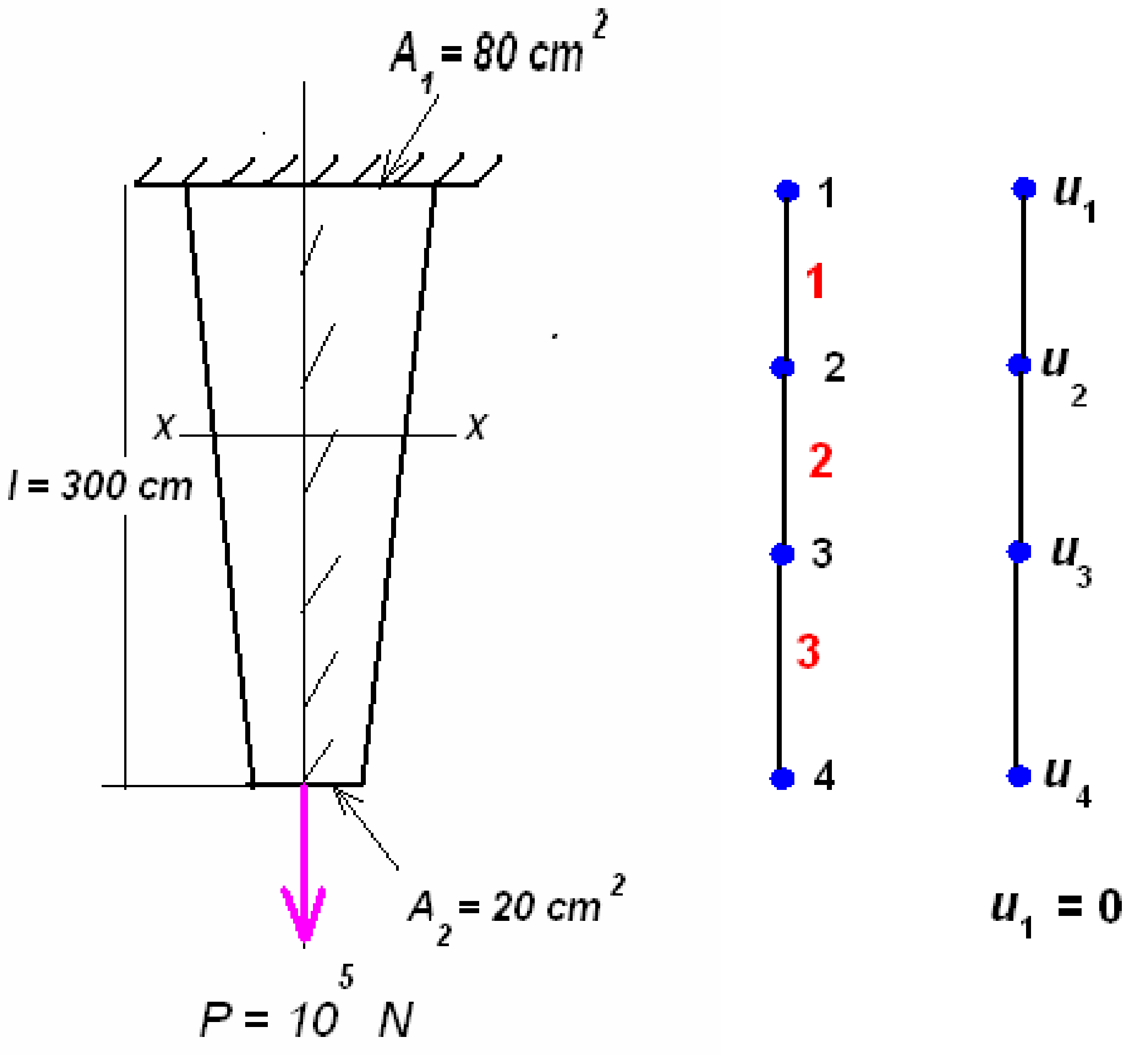
$$= \left\{\frac{\mathrm{A}_1}{6} l + \frac{\mathrm{A}_2}{3} l\right\}$$

Therefore the assembled load vector will be

$$\{r^e\} = \frac{\gamma l}{6}\begin{Bmatrix} 2A_1 + A_2 \\ 2A_2 + A_1 \end{Bmatrix}$$

Case - I: Discretize the Tapered Bar into 3 elements.

The length of each element $'l'$ = 100 cm.



A1 = 80 cm2
X
X
l = 300 cm
A2 = 20 cm2
P = 10 5 N
1
1
2
2
3
3
4
u1
u2
u3
u4
u1 = 0

$$K^1 = \frac{E}{l_1}\frac{\mathrm{A}_1 + \mathrm{A}_2}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{E}{100}\begin{bmatrix} 70 & -70 \\ -70 & 70 \end{bmatrix}$$

$$K^2 = \frac{E}{l_2}\frac{\mathrm{A}_2 + \mathrm{A}_3}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{E}{100}\begin{bmatrix} 50 & -50 \\ -50 & 50 \end{bmatrix}$$

$$K^3 = \frac{E}{l_3}\frac{\mathrm{A}_3 + \mathrm{A}_4}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{E}{100}\begin{bmatrix} 30 & -30 \\ -30 & 30 \end{bmatrix}$$

The global stiffness matrix will become

$$[K] = \begin{Bmatrix} [K^1] & & \\ & [K^2] & \\ & & [K^3] \end{Bmatrix}$$

$$= \frac{E}{100}\begin{bmatrix} 70 & -70 & & \\ -70 & 70+50 & -50 & \\ & -50 & 50+30 & -30 \\ & & -30 & 30 \end{bmatrix}$$

$$= \frac{E}{100}\begin{bmatrix} 70 & -70 & 0 & 0 \\ -70 & 120 & -50 & 0 \\ 0 & -50 & 80 & -30 \\ 0 & 0 & -30 & 30 \end{bmatrix}$$

$$\begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \frac{\gamma l}{6} \begin{Bmatrix} 2A_1 + A_2 \\ 2A_2 + A_1 \end{Bmatrix}$$

$$\{r^1\} = \gamma \times 100 \begin{Bmatrix} \dfrac{220}{6} \\ \dfrac{200}{6} \end{Bmatrix}$$

$$\{r^2\} = \gamma \times 100 \begin{Bmatrix} \dfrac{160}{6} \\ \dfrac{140}{6} \end{Bmatrix}$$

$$\{r^3\} = \gamma \times 100 \begin{Bmatrix} \dfrac{100}{6} \\ \dfrac{80}{6} \end{Bmatrix}$$

Similarly the assembled global load vector will become

$$[R] \quad = \quad \begin{Bmatrix} |\, r^1 \,| & & \\ & |\, \mathrm{r}^2 \,| & \\ & & |\, \mathrm{r}^3 \,| \end{Bmatrix} \quad + \quad \begin{Bmatrix} P^1 & & \\ & \mathrm{P}^2 & \\ & & \mathrm{P}^3 \end{Bmatrix}$$

The global load vector is

$$[R] = \gamma \times 100 \begin{Bmatrix} \frac{220}{6} \\ \frac{200}{6} + \frac{160}{6} \\ \frac{140}{6} + \frac{100}{6} \\ \frac{80}{6} \end{Bmatrix} + \begin{Bmatrix} R \\ O \\ O \\ P \end{Bmatrix}$$

$$= \frac{\gamma \times 100}{6} \begin{Bmatrix} 220 \\ 360 \\ 240 \\ 80 \end{Bmatrix} + \begin{Bmatrix} R \\ O \\ O \\ P \end{Bmatrix}$$

Now the total system of equation will be

$$\frac{E}{100}\begin{Bmatrix} 70 & -70 & & \\ -70 & 120 & -50 & \\ & 50 & 80 & -30 \\ & & -30 & 30 \end{Bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \frac{\gamma \ \text{x} \ 100}{6}\begin{Bmatrix} 220 \\ 360 \\ 240 \\ 80 \end{Bmatrix} \begin{Bmatrix} R \\ O \\ O \\ P \end{Bmatrix}$$

Now applying the Boundary conditions i.e. $u_1 = 0$ ..

Delete the first row and first column of elements and the system of equation will reduce to

$$\begin{Bmatrix} 120 & -50 & \\ -50 & 80 & -30 \\ & 30 & 30 \end{Bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \frac{\gamma \ \text{x} \ 100}{6}\begin{Bmatrix} 360 \\ 240 \\ 80 \end{Bmatrix} \begin{Bmatrix} O \\ O \\ P \end{Bmatrix}$$

The data are $E = 2 \times 10^7$ N/cm^2 $\gamma = 0.075$ N/cc and $P = 1 \times 10^5$ N.

On solving the above equation we get

$$u_4 = 0.035501997 \text{ cm}$$
$$u_3 = 0.018818567 \text{ cm}$$
$$u_2 = 0.008778557 \text{ cm}$$

The deflection at mid section of the bar by interpolation is

$$\mathbf{U_{x=50} = \frac{u_2 + u_3}{2}} = 0.01379856 \text{ cm}$$

Example 2 Let us consider the discretization with 2 elements

h = 150 cm

The assembled stiffness matrix will be

$$[K] = \frac{E}{150}\begin{bmatrix} 65 & -65 & \\ -65 & 65+35 & \\ & -35 & 35 \end{bmatrix}$$

Similarly the assembled load vector will be

$$[R] = \rho \times 150 \begin{Bmatrix} \frac{210}{6} & \\ \frac{180}{6} + & \frac{120}{6} \\ & \frac{90}{6} \end{Bmatrix} + \begin{Bmatrix} R \\ O \\ P \end{Bmatrix}$$

After applying the B.Cs the global system of equation will become

$$\frac{E}{150}\begin{bmatrix}100 & -35 \\ -35 & 35\end{bmatrix}\begin{Bmatrix}u_2 \\ u_3\end{Bmatrix} = \rho \text{ x } 150\begin{Bmatrix}\dfrac{240}{6} \\ \dfrac{80}{6}\end{Bmatrix} \begin{Bmatrix}O \\ P\end{Bmatrix}$$

On solving the above set of simultaneous equations we get

u_3 =0.033068406 cm (Tip displacement)

u_2 =0.011607692 cm (Mid section displacement)

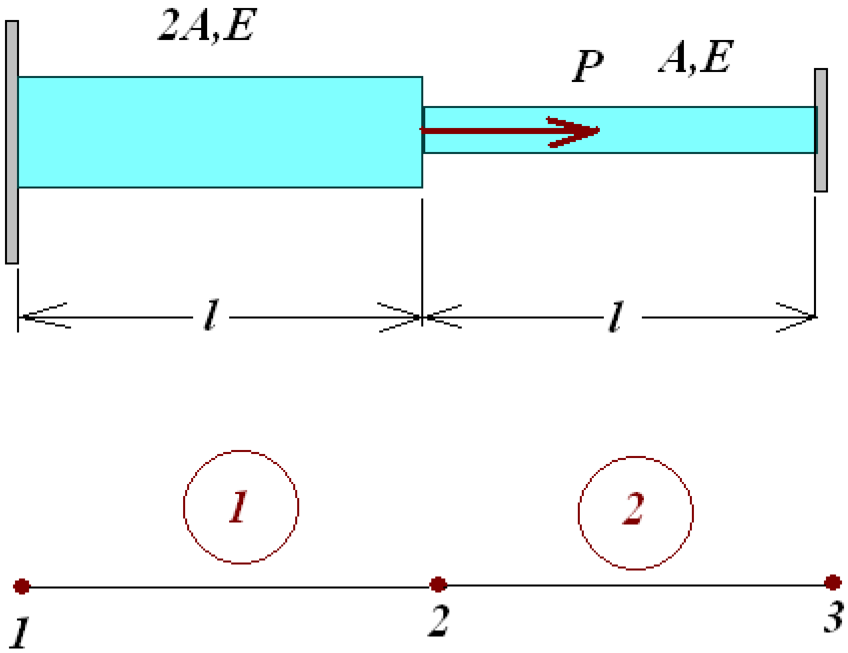
$$[K^e] = \frac{E}{l}\frac{A_1 + A_2}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For a bar of constant cross section $A_1 = A_2$

$$[K^e] = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{r^e\} = \frac{\gamma A l}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Example 3



Element 1,

$$\begin{array}{cc} & \begin{array}{cc} u_1 & u_2 \end{array} \\ \mathbf{k}_1 = \dfrac{2EA}{L} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Element 2,

$$\begin{array}{cc} & \begin{array}{cc} u_2 & u_3 \end{array} \\ \mathbf{k}_2 = \dfrac{EA}{L} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$\frac{EA}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Load and boundary conditions (BC) are,

$$u_1 = u_3 = 0, \qquad\qquad F_2 = P$$

FE equation becomes,

$$\frac{EA}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \end{Bmatrix}$$

Deleting the 1st row and column, and the 3rd row and column, we obtain,

$$\frac{EA}{L}[3]\{u_2\} = \{P\}$$

Thus,

$$u_2 = \frac{PL}{3EA}$$

Stress in element 1 is

$$\sigma_1 = E\varepsilon_1 = \quad E\frac{u_2 - u_1}{L}$$

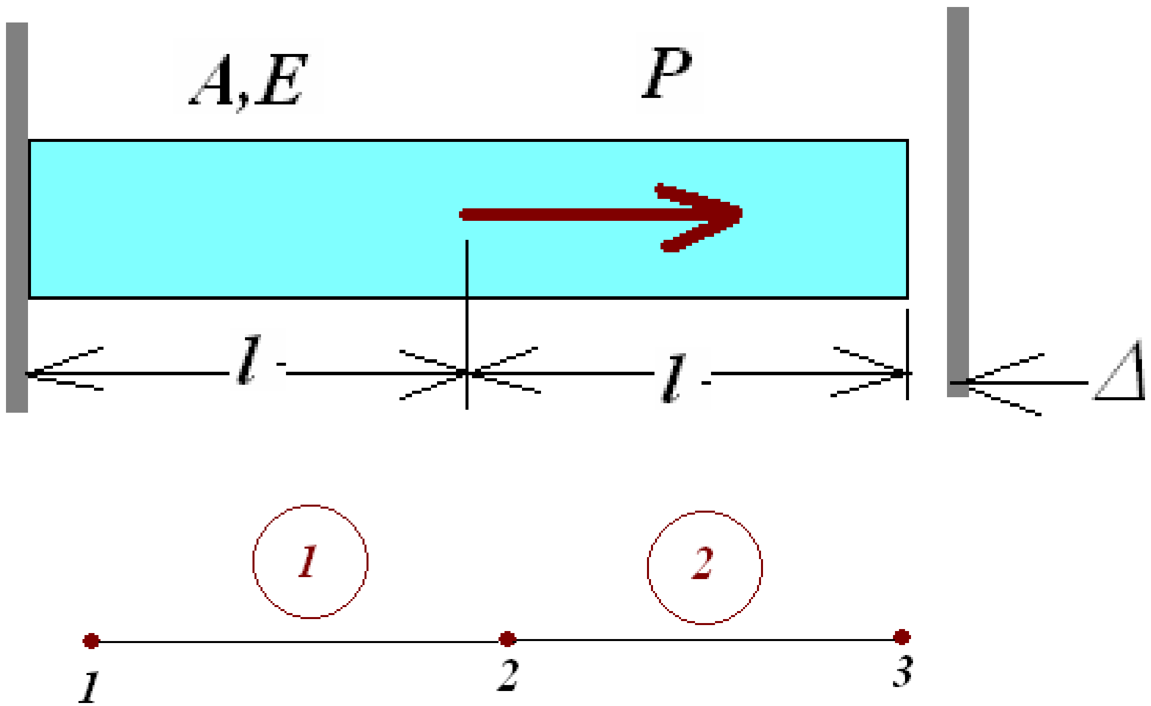
$$= \frac{E}{L}\left(\frac{PL}{3EA} - 0\right) \quad = \frac{P}{3A}$$

Similarly, stress in element 2 is

$$\sigma_2 = E\varepsilon_2 = \quad E\frac{u_3 - u_2}{L}$$

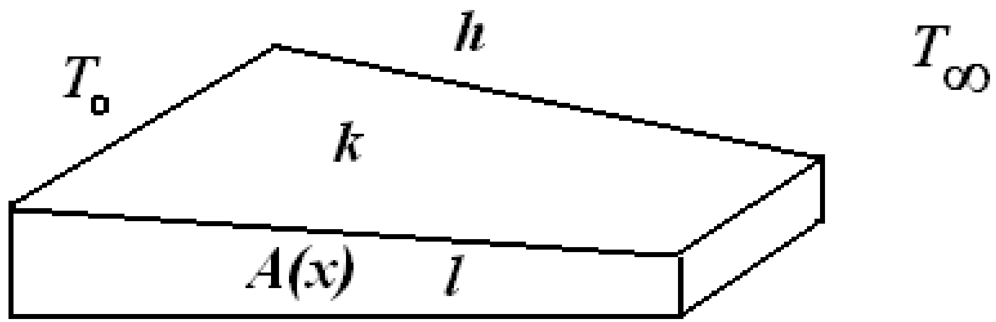
$$= \frac{E}{L}\left(0 - \frac{PL}{3EA}\right) = -\frac{P}{3A}$$

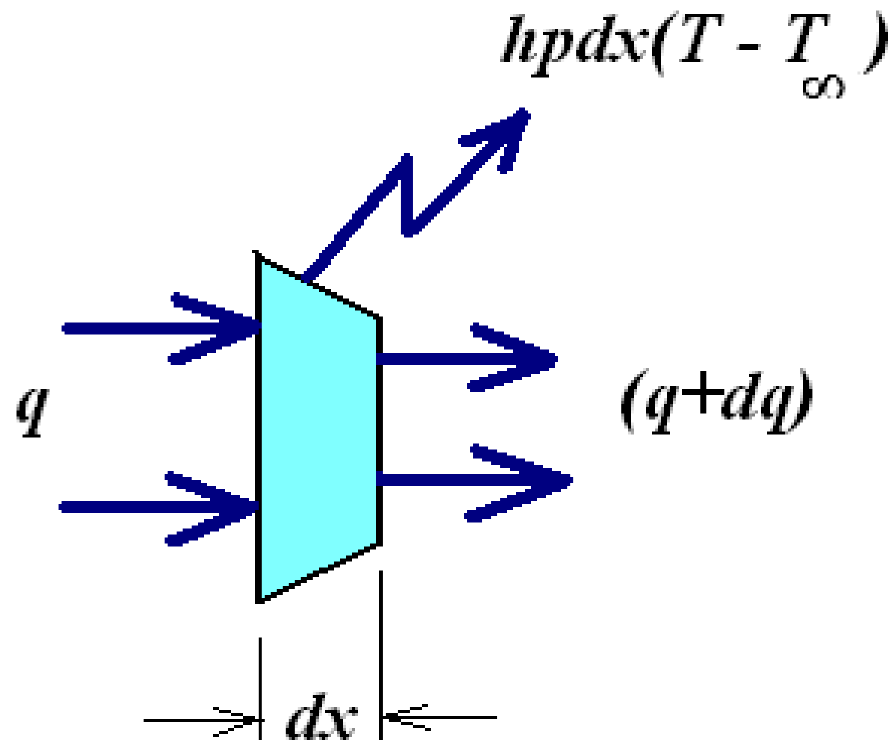
which indicates that bar 2 is in compression.



$$\frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ \Delta \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \end{Bmatrix}$$

WEAK FORM OF GOVERNING EQUATION FOR THERMAL PROBLEMS


T0
h
T∞
k
A(x)
l


hpdx(T - T∞)
q
(q+dq)
dx

where

k = Thermal conductivity coefficient

h = Thermal convection coefficient

A = Area of cross section subjected to CONDUCTION

p= Perimeter is the area exposed to CONVECTION

T_{∞} = Atmospheric Temp. , T = Variable

Q = Heat Source

$$(q + dq) - q + hp\, dx(T - T_{\infty}) = 0$$

$\div$ *by dx we get*

$$\frac{dq}{dx} + hp(T - T_{\infty}) = 0$$

$$\frac{d\left(-kA(x)\,\frac{dT}{dx}\right)}{dx} + hp(T - T_{\infty}) = 0$$

Boundary conditions:

i) At x= 0 $T = T_o$

ii) At the free end any one of the following three possible boundary conditions could be specified

1. If free end is insulated $-kA\, dT/dx = 0$
2. If free end is open to atmosphere
   $-kA\, dT/dx|_{=l} = hA(T - T_\infty)$
3. Specified temperature $T(l) = T_l$

The governing equation for heat transfer in a one dimensional problem is given by

$$\frac{d}{dx}\left[-KA\frac{dT}{dx}\right]+hp(T-T_{\infty})=0$$

The weak form can be obtained by

$$\int w(x)R(x)dx=0$$

For a bar of length ‘l’ with wall temperature ‘T’ the weak form of the governing equation becomes

$$\int_0^l w(x)\left[\frac{d}{dx}\left[-KA\frac{dT}{dx}\right]+hp(T-T_\infty)\right]dx=0$$

$$\int_0^l w(x)\frac{d}{dx}\left[-KA\frac{dT}{dx}\right]dx+\int_0^l w(x)hp(T-T_\infty)dx=0$$

1

Let $$I_1=\int_0^l w(x)\frac{d}{dx}\left[-KA\frac{dT}{dx}\right]dx$$

and $u=w(x)$ $\qquad du=dw$

$$dv=\frac{d}{dx}\left[-KA\frac{dT}{dx}\right]dx \qquad v=-KA\frac{dT}{dx}$$

$$I_1 = uv - \int v\,du$$

$$I_1 = w(x)\left[-KA\frac{dT}{dx}\right]_0^l - \int_0^l \left[-KA\frac{dT}{dx}\right]\frac{dw}{dx}dx$$

Substituting the above term in equation 1, we get

$$w(x)\left[-KA\frac{dT}{dx}\right]_0^l - \int_0^l \left[-KA\frac{dT}{dx}\right]\frac{dw}{dx}dx + \int_0^l w(x)hp(T - T_\infty)dx = 0$$

$$\underbrace{w(x)\left[-KA\frac{dT}{dx}\right]_0^l}_{\text{Boundary term}} + \underbrace{\int_0^l KA\frac{dT}{dx}\frac{dw}{dx}dx}_{B_1(T,w)} + \underbrace{\int_0^l hpw(x)T(x)dx}_{B_2(T,w)} - \underbrace{\int_0^l hpw(x)T_\infty dx}_{l(w)} = 0$$

$$\int_0^l KA\frac{dT}{dx}\frac{dw}{dx}dx + \int_0^l hpw(x)T(x)dx = \int_0^l hpw(x)T_\infty dx - w(x)\left[hA(T_L - T_\infty)\right]$$

Substituting in the weak form

$\boldsymbol{T(x) = N_1 T_1 + N_2 T_2}$

And w(x) as $\boldsymbol{N_1}$ first and then $\boldsymbol{N_2}$ we get a system of two equations in two unknowns namely T_1 and T_2 which can be written as

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}_{cond} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}_{conv} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

Where

$$K^e_{ij_{cond}} = \int_0^l kA(x) \frac{dN_i}{dx} \frac{dN_j}{dx} dx$$

$$K^e_{ij_{conv}} = \int_0^l hp(x) \, N_i N_j dx$$

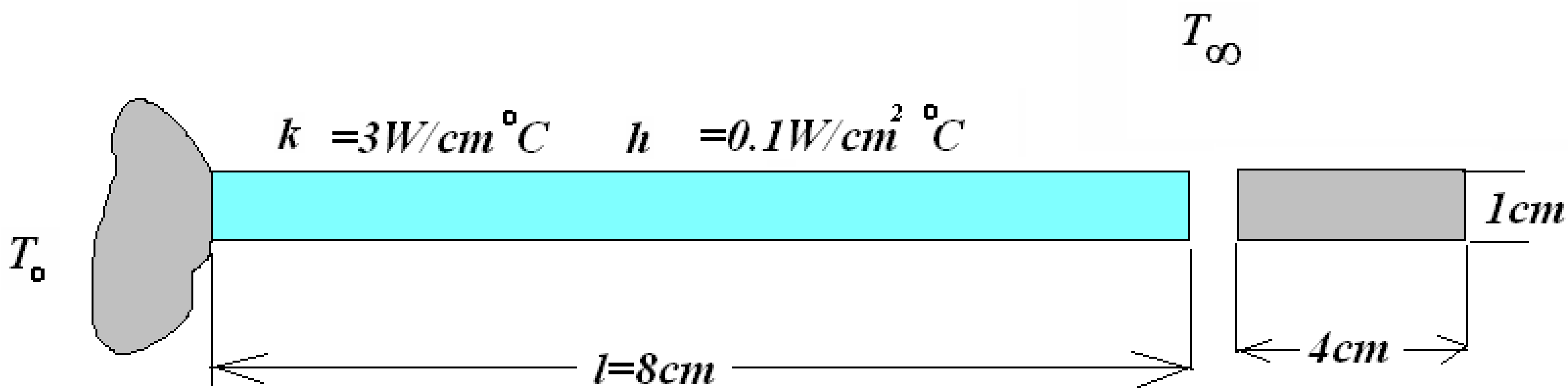
$$q^e_j = \int_0^l hpT_\infty N_j \, dx$$

Let the elements be of equal length l

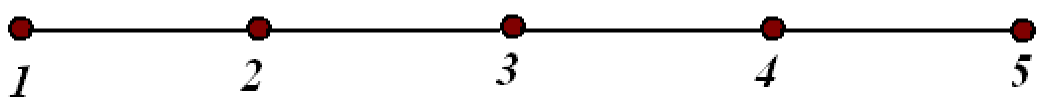
The element matrices are

$$[K^e] = \frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hP\,l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & hA \end{bmatrix}$$

$$[f^e] = \frac{hPl\,T_\infty}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ hA\,T_\infty \end{Bmatrix}$$



T_∞
k =3W/cm °C
h =0.1W/cm^2 °C
T_o
1cm
l=8cm
4cm



1
2
3
4
5

Boundary conditions:

at x = 0, T(0) = T

at x = L, $-KA \dfrac{dT}{dx} \Big|_{l} = hA\,(T_l - T_\infty)$

conduction = convection loss

For a typical linear element

$$N_I = 1 - (x/l)$$

$$N_J = (x/l)$$

Let the elements be of equal length $l = 2\ cm$

The element matrices are

$$[K^e] = \frac{kA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hp\,l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & hA \end{bmatrix}$$

$$[q^e] = \frac{hpl\ \ T_\infty}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ hA\,T_\infty \end{Bmatrix}$$

The element matrices for ELEMENT (1), (2) & (3) are

$$[K^e]_{cond} = \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \; ; \; \{q_e\} = \begin{Bmatrix} 20 \\ 20 \end{Bmatrix}$$

$$[K^e]_{conv} = \begin{bmatrix} 0.667 & 0.333 \\ 0.333 & 0.667 \end{bmatrix} \; ; \; \{q_e\} = \begin{Bmatrix} 20 \\ 20 \end{Bmatrix}$$

$$[K^e]_{therm} = \begin{bmatrix} 6.666 & -5.667 \\ -5.667 & 6.666 \end{bmatrix} \; ; \; \{q_e\} = \begin{Bmatrix} 20 \\ 20 \end{Bmatrix}$$

The element matrix for ELEMENT (4) is

$$[K^e]_{cond} = \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}$$

$$[K^e]_{conv} = \begin{bmatrix} 0.667 & -0.333 \\ -0.333 & 0.667 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0.4 \end{bmatrix}$$

$$\{q_e\} = \begin{Bmatrix} 20 \\ 20 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 8 \end{Bmatrix}$$

$$[K^e]_{therm} = \begin{bmatrix} 6.666 & -5.667 \\ -5.667 & 7.066 \end{bmatrix} \; ; \; \{q_e\} = \begin{Bmatrix} 20 \\ 28 \end{Bmatrix}$$

On assembly we get

$$\begin{bmatrix} 6.667 & -5.667 & 0 & 0 & 0 \\ -5.667 & 13.33 & -5.667 & 0 & 0 \\ 0 & -5.667 & 13.33 & -5.667 & 0 \\ 0 & 0 & -5.667 & 13.33 & -5.667 \\ 0 & 0 & 0 & -5.667 & 7.066 \end{bmatrix} * \begin{Bmatrix} T1 \\ T2 \\ T3 \\ T4 \\ T5 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 20+20 \\ 20+20 \\ 20+20 \\ 28 \end{Bmatrix}$$

By applying Boundary condition at

at x = 0 T = T_0 = 80º

$$\begin{bmatrix} 13.33 & -5.667 & 0 & 0 \\ -5.667 & 13.33 & -5.667 & 0 \\ 0 & -5.667 & 13.33 & -5.667 \\ 0 & 0 & -5.667 & 7.066 \end{bmatrix} * \begin{Bmatrix} T2 \\ T3 \\ T4 \\ T5 \end{Bmatrix} = \begin{Bmatrix} 40 + 5.667 * 80 \\ 40 \\ 40 \\ 28 \end{Bmatrix}$$

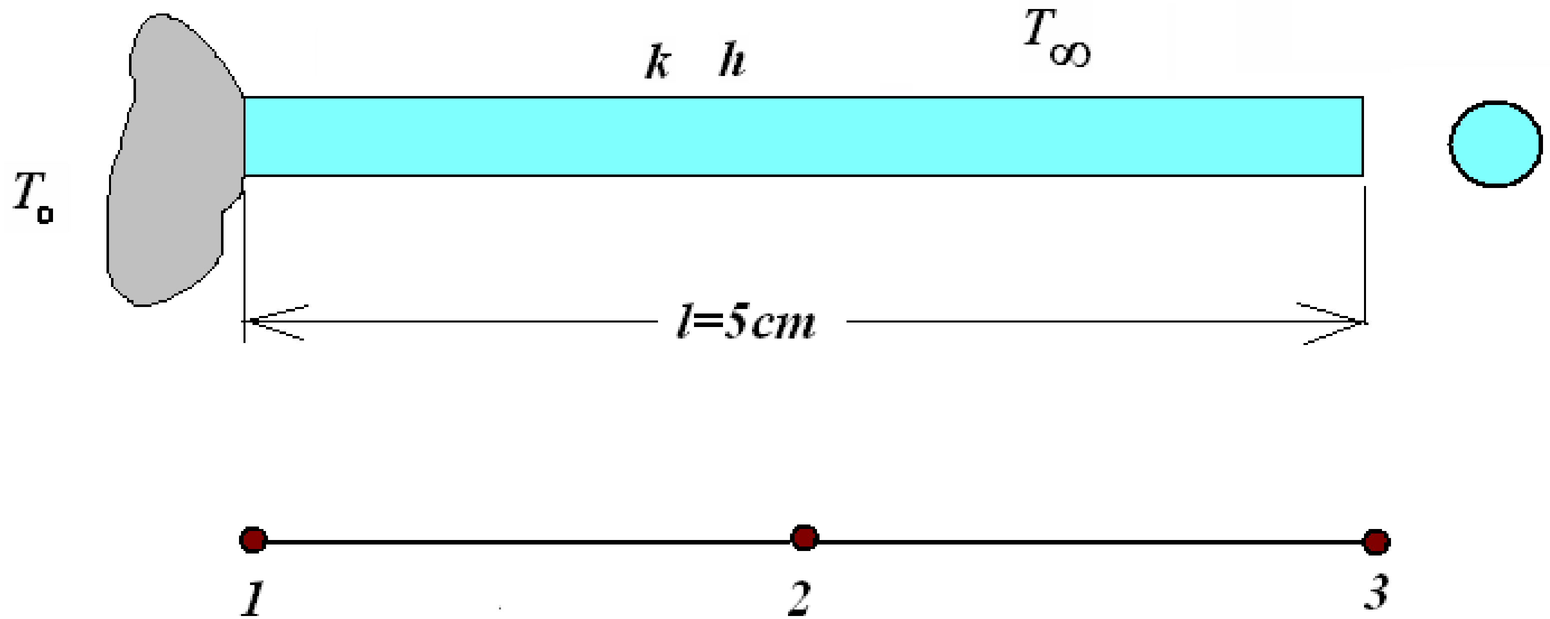
By solving we get

$T_2 = 53.95^0 \ C;$

$T_3 = 39.88^0 \ C;$

$T_4 = 32.82^0 \ C;$

$T_5 = 30.29^0 \ C;$



Boundary condition: Free end insulated

$h = 10\ W/cm^2\ {}^oC$

$K = 70\ W/cm\ {}^oC$

$T_0 = 140^oC$

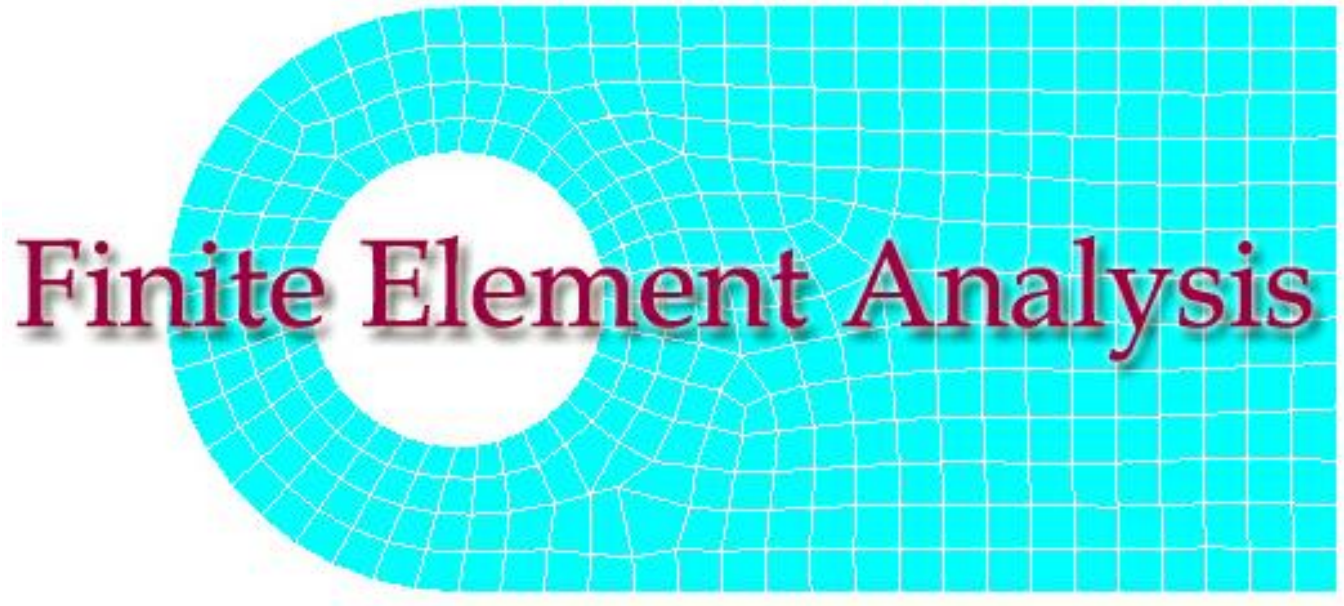
$T_\infty = 40^oC$

$\ell = 5\ cm$

Radius $r = 1\ cm$

Area $A = \pi\ r2 = \pi\ cm2$

Perimeter $p = 2\pi r = 2\ \pi$



LECTURE 5

1

We have seen so far the application of the two noded linear element to the following applications

- ➢ Structural problems
- ➢ ID heat transfer through fins

$$N_1(x) = 1 - x/\ell \qquad N_1(0) = 1. \quad N_1(\ell) = 0$$

$$N_1(x) = x/\ell \qquad N_2(0) = 0. \quad N_2(\ell) = 1$$

$$N_1 + N_2 = 1$$

Structural problems

The governing equation is

$$\frac{d}{dx}[EA(x)\ \frac{du}{dx}] + \gamma A(x) = 0 \qquad \text{in } 0 < x < l$$

With B.Cs i) $u(0) = 0$

and

ii)At $x=l$ $\quad [EA(x)\ \frac{du}{dx}] = P$

Weak form is given by

$$\int_0^L EA(x)\frac{du}{dx}\frac{dw}{dx}\,dx = \int_0^L \gamma\, A(x)\, w(x)\, dx + P(L)w(L) - P(0)\, w(0)$$

$$[K^e] = \frac{E}{l}\frac{A_1 + A_2}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{r^e\} = \frac{\gamma l}{6}\begin{Bmatrix} 2A_1 + A_2 \\ 2A_2 + A_1 \end{Bmatrix}$$

For a bar of constant cross section $A_1 = A_2$

$$[K^e] = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{r^e\} = \frac{\gamma A l}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

ID heat transfer through fins

$$\frac{d}{dx}\left[-KA\frac{dT}{dx}\right]+hp(T-T_{\infty})=0$$

$$\int_0^l KA\frac{dT}{dx}\frac{dw}{dx}dx+\int_0^l hpw(x)T(x)dx=$$

$$\int_0^l hpw(x)T_{\infty}dx-w(x)\left[hA(T_L-T_{\infty})\right]$$

The element matrices are

$$[K^e] = \frac{kA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hp\,l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & hA \end{bmatrix}$$

$$[f^e] = \frac{hpl\,T_\infty}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ hA\,T_\infty \end{Bmatrix}$$

LONGITUDINAL VIBRATION

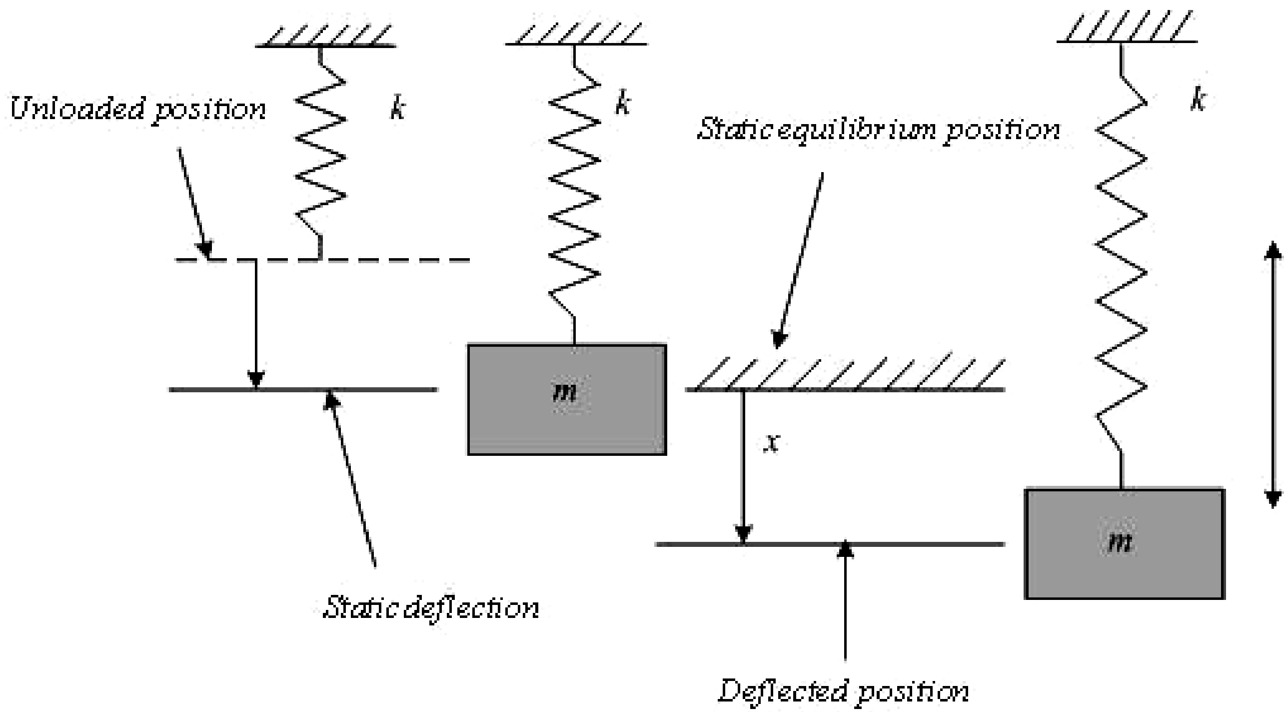
What is vibration?

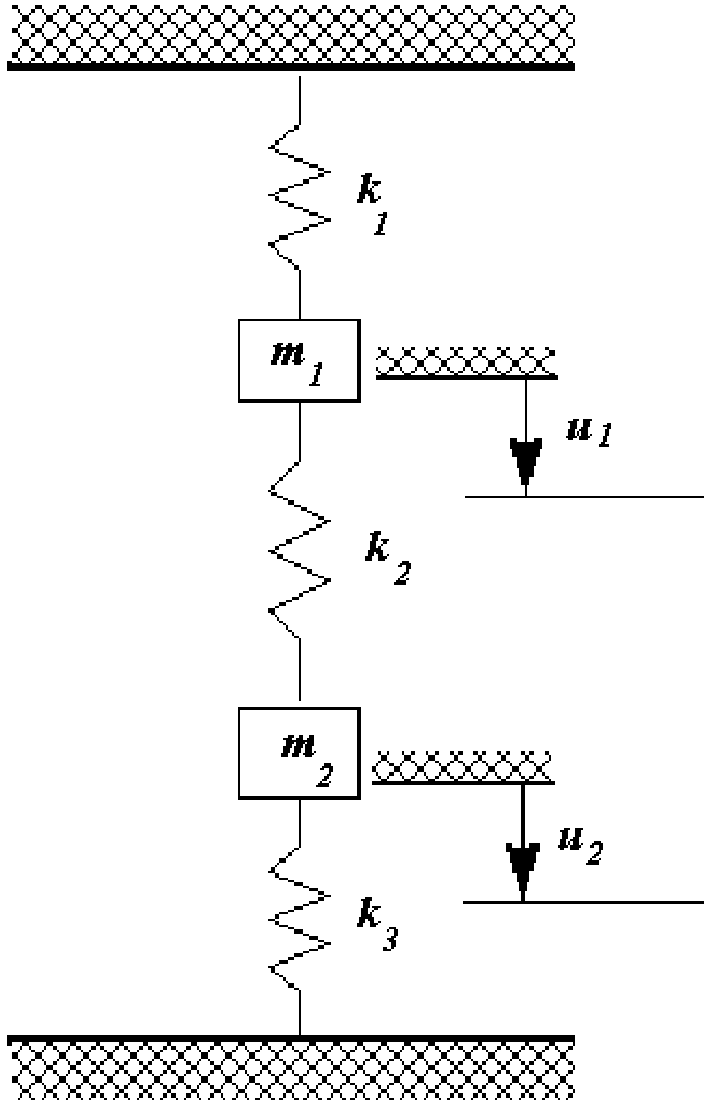
What is natural frequency?

What is meant by degree of freedom of a vibrating body?

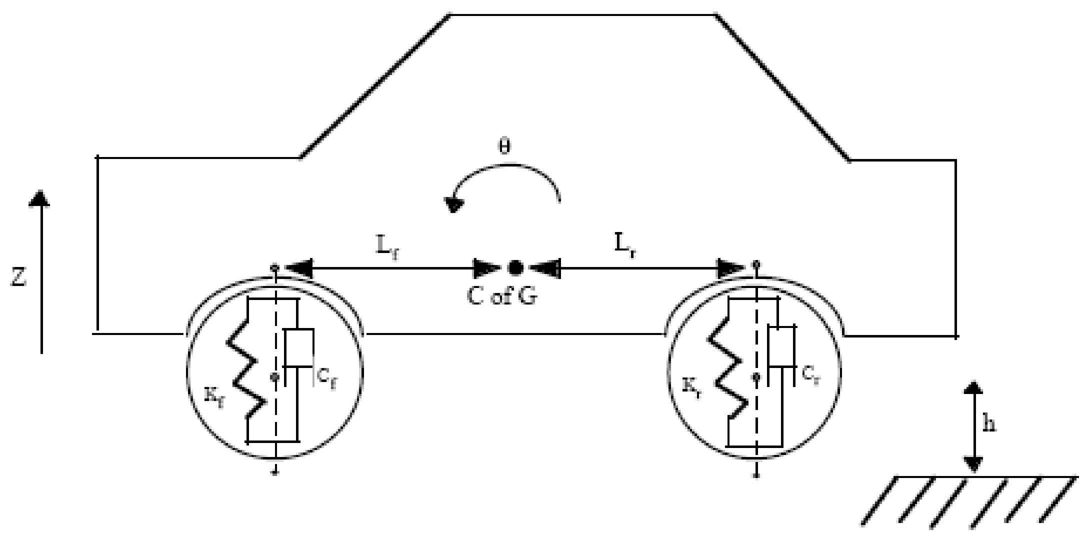
What is free vibration?

What is forced vibration?

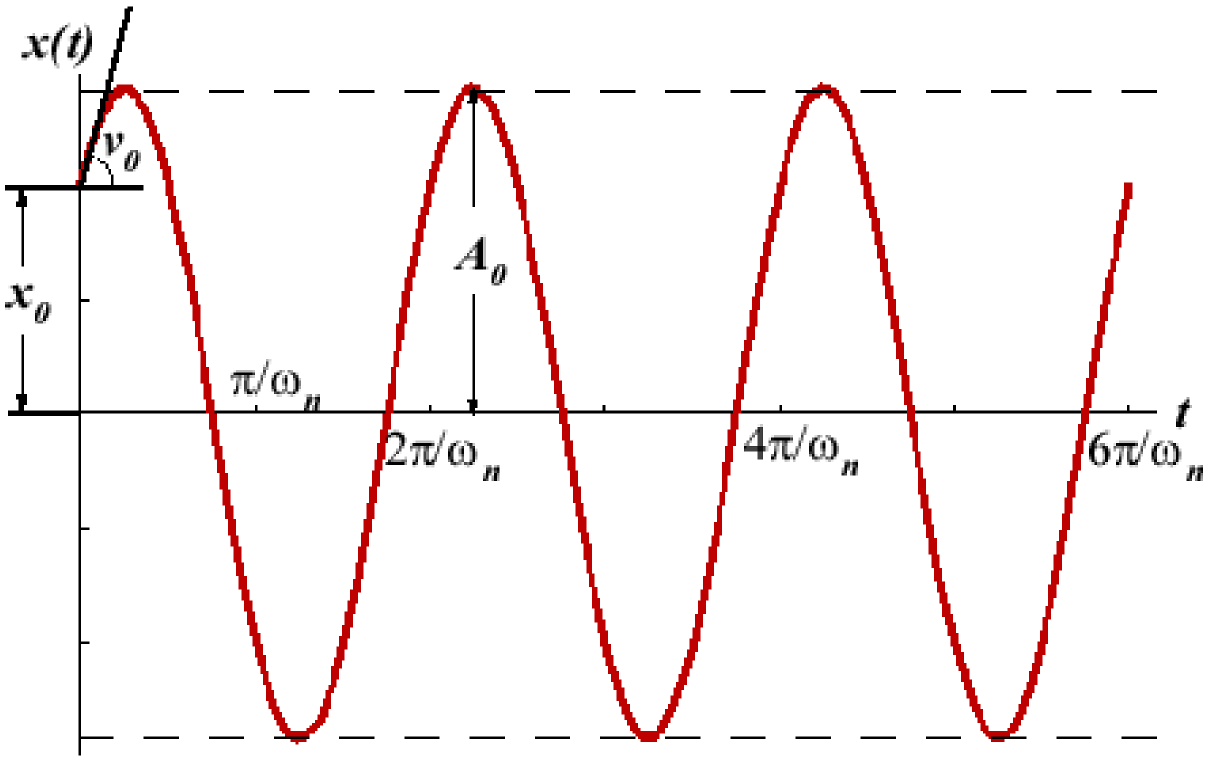

Unloaded position
k
k
Static equilibrium position
k
m
x
m
Static deflection
Deflected position



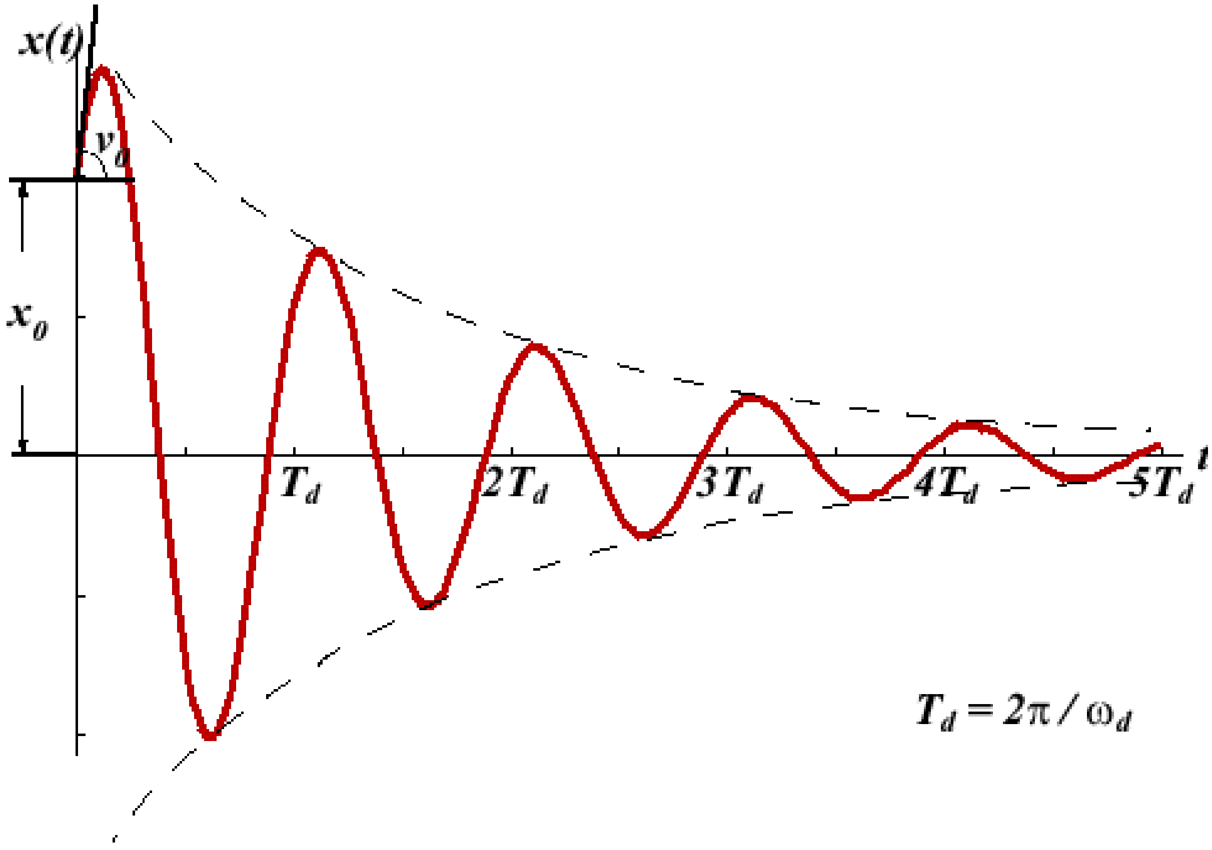
k
1
m
1
u 1
k
2
m
2
u 2
k
3

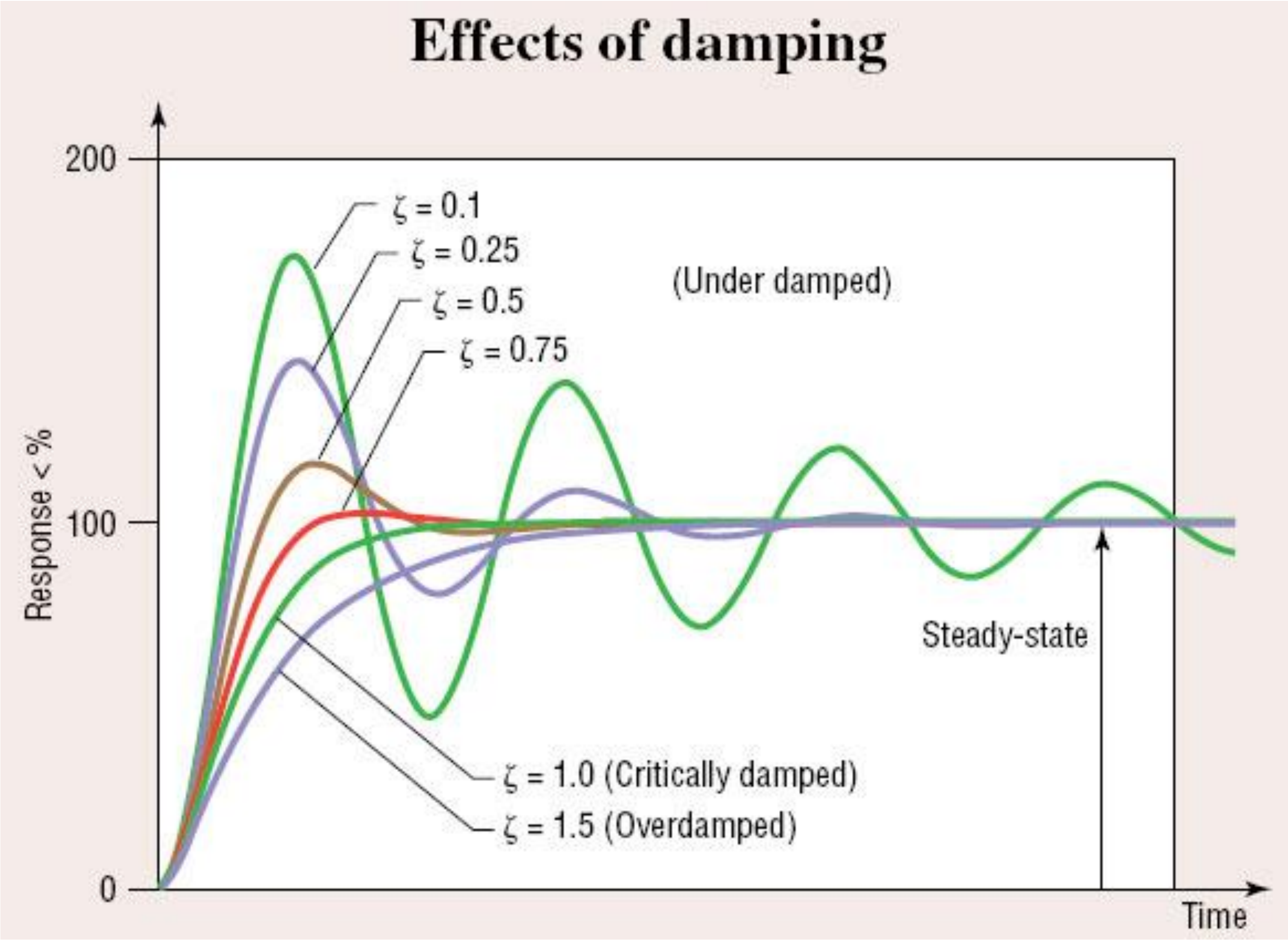

θ
Z
L_f
L_r
C of G
C_f
K_f
K_r
C_r
h

Free undamped vibration



Free damped vibration





Effects of damping
200
100
0
Response < %
ζ = 0.1
ζ = 0.25
ζ = 0.5
ζ = 0.75
(Under damped)
Steady-state
ζ = 1.0 (Critically damped)
ζ = 1.5 (Overdamped)
Time

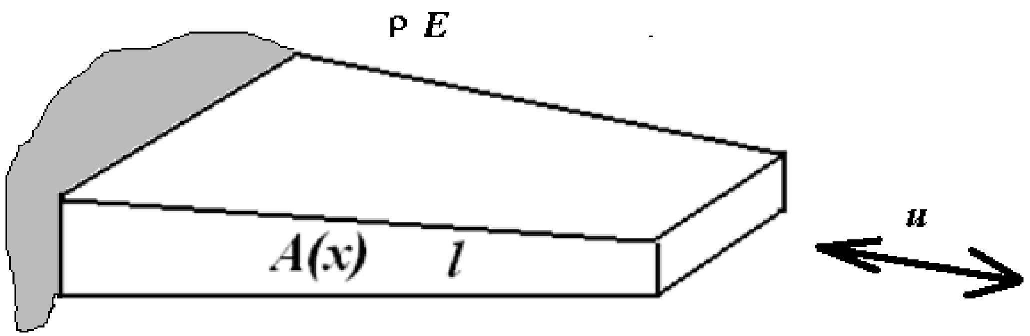
Longitudinal Vibrations of Elastic Rod:

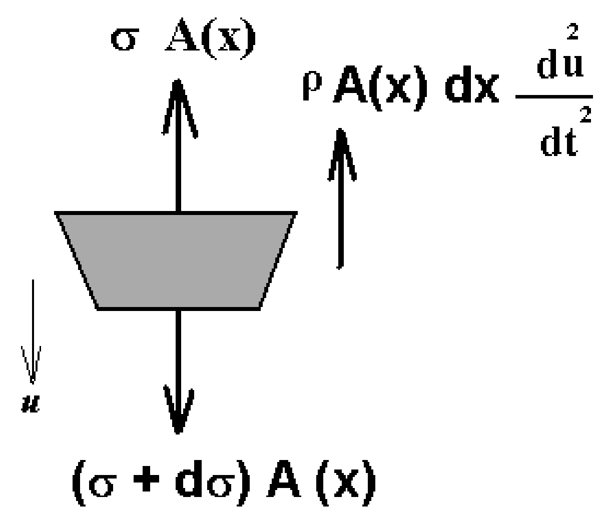
$\therefore \sigma A(x) - (\sigma + d\sigma) A(x) + \text{I.F.} = 0$ --- $\rightarrow$(1)

ie., $d\sigma A(x) - IF = 0$

We know that IF is given by product of mass and acceleration.

$$\text{Acceleration} = \frac{d^2u}{d\,t^2}$$





$$IF = m \times a = (\rho . A(x)dx)\, d^2u/\, dt^2$$

$$= \rho A(x)\, dx . \ddot{u}$$

Substituting in equation (1) we get

$$d\sigma A(x) - \rho A(x)\, dx .\ \ddot{u} = 0$$

or

$$\frac{d\sigma A(x)}{dx} - \rho A(x)\, \ddot{u} = 0$$

Now $\sigma = E\varepsilon = E\ \ du/dx$

$$\therefore \frac{d}{dx}\, EA(x)\, \frac{du}{dx} - \rho A(x)\, \ddot{u} = 0 \text{ ---} \qquad \rightarrow (2)$$

Assume that the displacement u is given by a harmonic function namely

$u = U \sin \omega_n t$

Velocity $= \acute{u} = \frac{du}{dt} = U\, \omega_n \cos \omega_n t$

Acceleration $\ddot{u} = \frac{d^2u}{dt^2} = -\, U\, \omega_n^2 \sin \omega_n t$

$$= -\, u\, \omega_n^2 \qquad \rightarrow (3)$$

$$\frac{d}{dx}\left(E\,A(x)\,\frac{du}{dx}\right) + \rho.A.u\,\omega_n^2 = 0 \qquad (4)$$

For a bar fixed at one end the Boundary conditions are

i) $u(0) = 0$

ii) $EA(x)\frac{du}{dx}$ at $x=l = 0$

$$\frac{d}{dx}\left[EA\frac{du}{dx}\right]+\rho A(x)u\omega_n^{\ 2}=0$$

$$\int\left(\frac{d}{dx}\left[EA\frac{du}{dx}\right]+\rho A(x)u\omega_n^{\ 2}\right)v(x)dx=0$$

$$-\int_0^l EA(x)\,\frac{du}{dx}\,\frac{dv}{dx}\,dx + \int_0^l \rho A(x)u(x)\,v(x)\,dx\omega_n{}^2$$

$$+ P(l)v(l) - P(0)\,v(0) = 0$$

P(l) = 0 and v(0) =0 $\therefore$ Weak form becomes

$$\int_0^l EA(x)\,\frac{du}{dx}\,\frac{dv}{dx}\,dx - \int_0^l \rho A(x)u(x)\,v(x)\,dx\omega_n{}^2 = 0$$

$$\int_0^l EA(x)\,\frac{du}{dx}\,\frac{dv}{dx}\,dx - \int_0^l \rho A(x) u(x)\, v(x)\, dx \omega_n^{\,2} = 0$$

Substituting in the weak form

$\boldsymbol{u(x) = N_1 u_1 + N_2 u_2}$

And v(x) as $\boldsymbol{N_1}$ first and then $\boldsymbol{N_2}$ we get a system of two equations in two unknowns namely u_1 and u_2 which can be written as

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \omega_{\mathrm{n}}^{\ 2} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

$$K_{ij}^{e} = \int_0^l EA(x) \frac{dN_i}{dx} \frac{dN_j}{dx} dx$$

$$M_{ij}^{e} = \int_0^l \rho A(x) N_i N_j dx$$

$$[K^e] = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[M^e] = \frac{\rho A\, l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\left[\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \omega_{\mathrm{n}}{}^{2}\right]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

$$\left[\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \omega_{\mathrm{n}}^{2}\right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

Here ω_n represents the natural frequency or eigen value and the vector of unknown displacements represents the eigen vector associated with each eigen value

$$\left[\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \omega_{\mathrm{n}}{}^{2}\right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

$$\left[\mathrm{K} - \mathrm{M}\omega_{\mathrm{n}}{}^{2}\right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

Since {u} which represents the vector of nodal displacements, is not zero $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \neq 0$

$$\left|\mathrm{K} - \mathrm{M}\omega_{\mathrm{n}}{}^{2}\right| = 0$$

Which gives a quadratic in λ, where $\lambda = \omega_n^2$

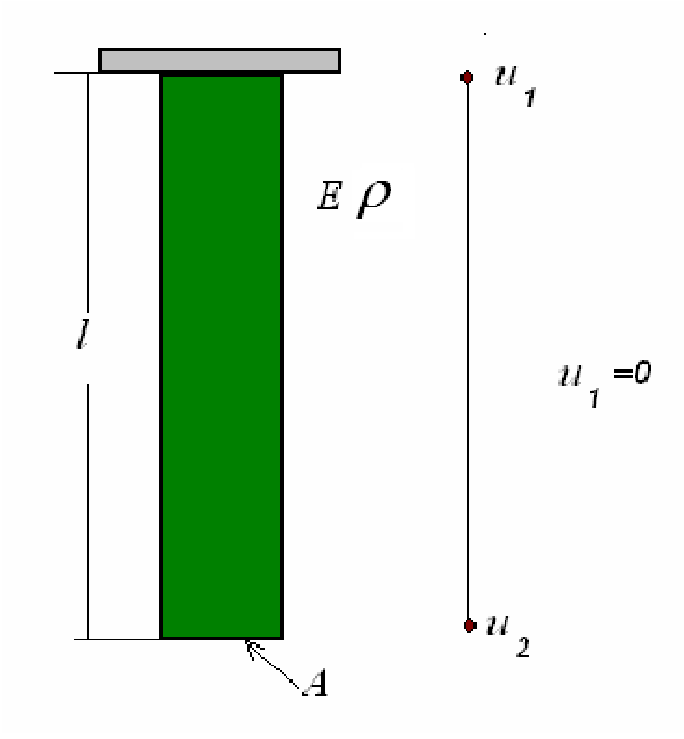
Solving for λ we get the eigen values

$$\left[\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \omega_{\mathrm{n}}{}^{2}\right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

Substituting ω_n^2 in the above eqn we get the vector of unknown displacements

Example - 1 Longitudinal Vibrations of Elastic Road

Consider a bar of cross – sectional area A and length `ℓ' fixed at one end and subjected to longitudinal vibration. We can model the bar using one two noded linear element.



Governing equation is

$$\frac{d}{dx}\left[EA\frac{dU}{dx}\right] - \omega_n{}^2 \rho A u = 0$$

The stiffness & mass matrices are respectively given by

$$[K] = \frac{EA}{\ell}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad [M] = \frac{PA\ell}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

The equilibrium equation is given by

$$\left([K] - \{M] \omega_n^2\right) \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

$$\text{i.e} \left(\frac{EA}{\ell} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} - \frac{\rho A \ell}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \omega^2{}_n \right) \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

or

$$\left(\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} - \frac{\rho \ell^2}{6E} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \omega^2{}_n \right) \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

As $u_1 = 0$ the above equation reduces to

$$\left(\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} - \frac{\rho \ell^2}{6E} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \omega^2_n \right) \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

As $u_2 \neq 0$ $\quad 1 - \dfrac{\rho \ell^2}{3E} \omega^2_n = 0$

or

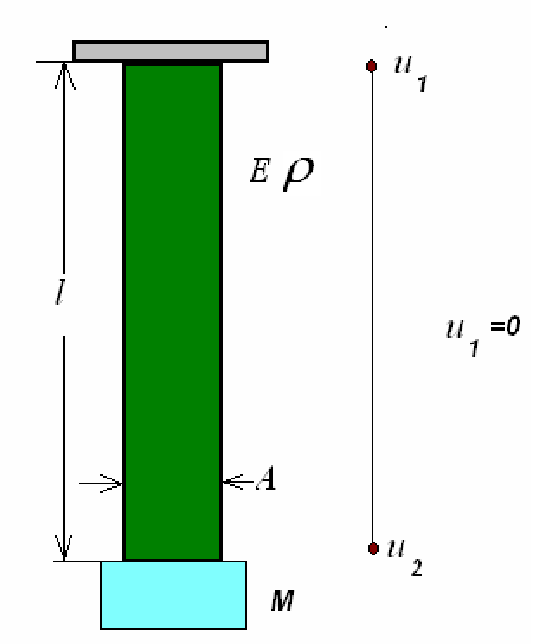
$$\omega_n = \frac{\sqrt{3}}{\ell} \sqrt{\frac{E}{\rho}} = \frac{1.732}{l} \sqrt{\frac{E}{\rho}}$$

Example 2: Now we shall see the effect of a concentrated mass "M" at the end of the bar

$$[K] = \frac{EA}{\ell}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$[M] = \frac{\rho A\ell}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}$$

$$[M] = \frac{\rho A\ell}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 + \frac{M \times 6}{\rho A\ell} \end{pmatrix}$$



Applying the Boundary condition that $u_1 = 0$ we get

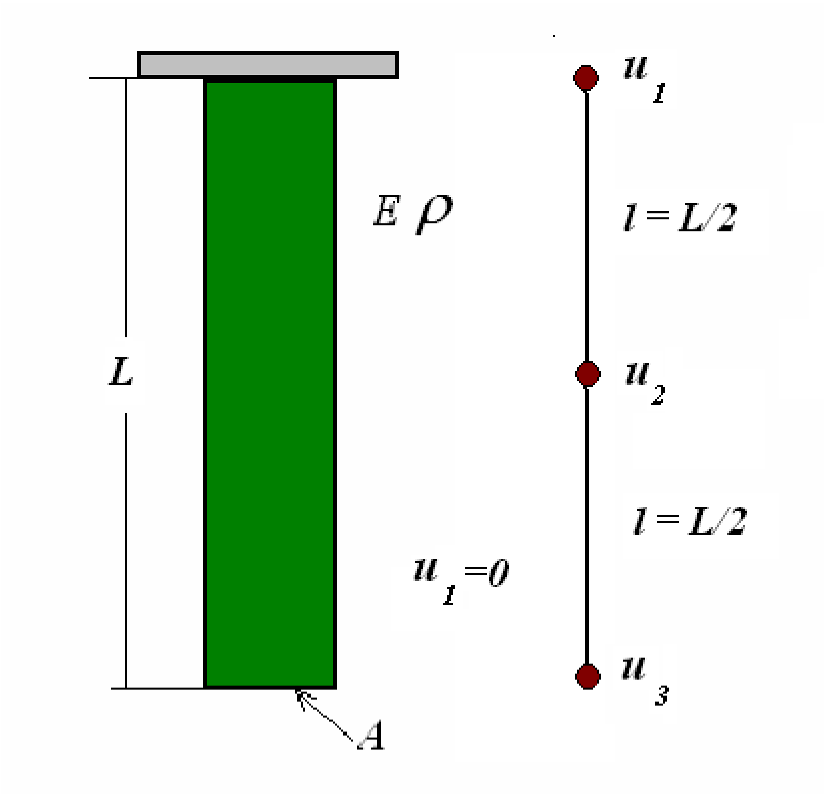
$$1 - \frac{\rho A \ell^2}{3E}\omega_n^2 - \frac{\rho A \ell}{6} \times \frac{6M}{\rho A \ell}\omega_n^2 = 0$$

or

$$\omega_n^2 \left(\frac{\rho A \ell^2}{3E} + M \right) = 1$$

$$\therefore \omega_n = \frac{1}{\sqrt{\frac{\rho A \ell^2}{3E} + M}}$$

Example 3: Consider the same bar fixed at one end and subjected to longitudinal vibration. Divide the bar into two elements of length ***l***



Elemental matrices are given by

$$[K]^1 = [K]^2 = \frac{2EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[M]^1 = [M]^2 = \frac{\rho AL}{12} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Global matrices are

$$[K] = \frac{2EA}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[M] = \frac{\rho AL}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad \{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The equation is [K] {u} - α[M] {u}=0

The boundary condition is $u_1=0$ The reduced equation is

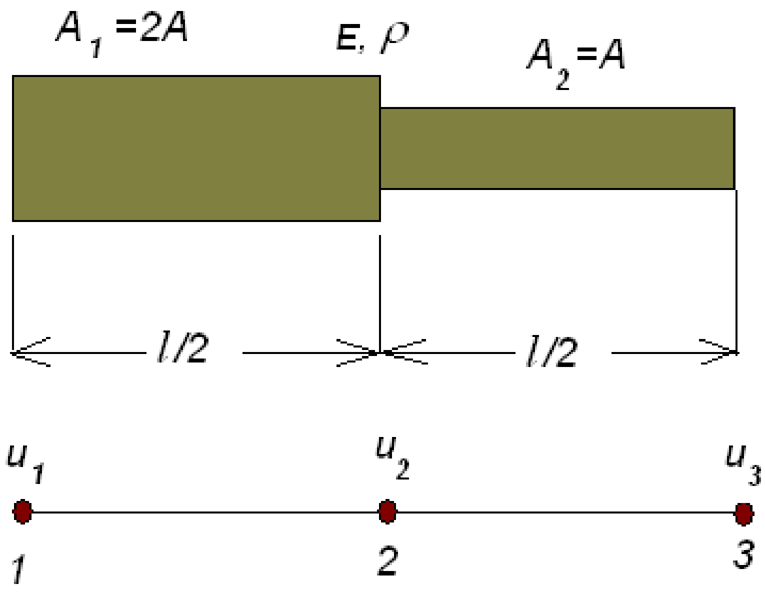
$$\begin{vmatrix} (2-4\alpha) & (-1-\alpha) \\ (-1-\alpha) & (1-2\alpha) \end{vmatrix} = 0$$

when $\alpha = \dfrac{L^2}{24}\dfrac{\rho}{E}\omega_n^{\ 2}$

The natural frequencies are $\omega_1 = \dfrac{2.33}{L}\sqrt{\dfrac{E}{\rho}}$ and

$$\omega_2 = \frac{3.88}{L}\sqrt{\frac{E}{\rho}}$$

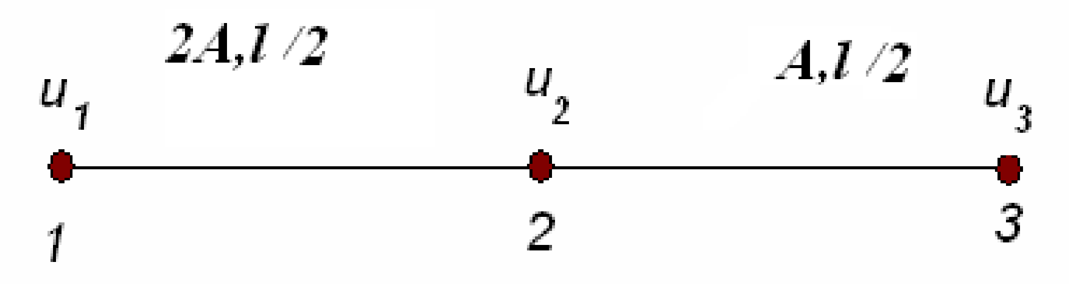
Example 4:- Determine the natural frequencies of longitudinal vibration of the unconstrained stepped bar shown in Fig.



The Stiffness & mass matrices of the two elements are given by

$$[K]^1 = \frac{A_1E_1}{\ell_1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{4AE}{l}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$[K]^2 = \frac{A_2E_2}{\ell_2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{2AE}{l}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$



$$[M]^1 = \frac{\rho A_1 \ell_1}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \frac{\rho A l}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$[M]^2 = \frac{\rho A_2 \ell_2}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \frac{\rho A l}{12}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

The assembled stiffness & mass matrices are given by

$$[K]^g = \frac{2AE}{l}\begin{pmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$[M]^g = \frac{\rho A l}{12}\begin{pmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

The bar is unconstrained So the boundary conditions involve only specification of forces at the ends of the bar i.e.

$$EA\frac{du}{dx} = 0 \text{ at } x = 0 \text{ \& } x = l$$

The frequency equation can now be written as

$$\left| \frac{2AE}{l}\begin{pmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix} - \omega^2_n \frac{\rho A l}{12}\begin{pmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{pmatrix} \right| = 0$$

Dividing throughout by $\frac{2AE}{l}$ & defining $\frac{\rho l^2 \omega^2_n}{24E}$ as λ

We get

$$\begin{vmatrix} 2(1-2\lambda) & -2(1+\lambda) & 0 \\ -2(1+\lambda) & 3(1-2\lambda) & -(1+\lambda) \\ 0 & -(1+\lambda) & (1-2\lambda) \end{vmatrix} = 0$$

The evaluation of the determinant yields

$18\lambda(1-2\lambda)(\lambda-2) = 0$

The roots of the above equation gives the natural frequencies of the bar as

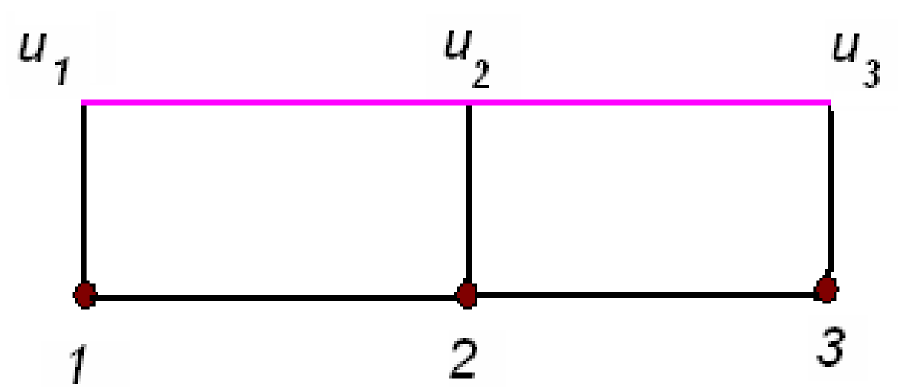
$\lambda = 0$ or $\omega_n = 0$ [Rigid Body Displacement]

$\lambda = ½$ or $\omega_{n_1} = \frac{3.46}{l}\sqrt{\left(\frac{E}{\rho}\right)}$ [First Natural Frequency]

$\lambda = 2$ or $\omega_{n_2} = \frac{6.92}{l}\sqrt{\left(\frac{E}{\rho}\right)}$ [Second Natural Frequency]

The first frequency $\omega_n = 0$, corresponds to the condition where all parts of the bar are subjected to equal displacements and hence it is unstressed. It represents rigid body mode shape for which the eigen vector is given by

$$\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$



The 2nd and 3rd frequencies correspond to elastic deformation modes and to determine the mode shape corresponding to these 2 frequencies we solve for the equations

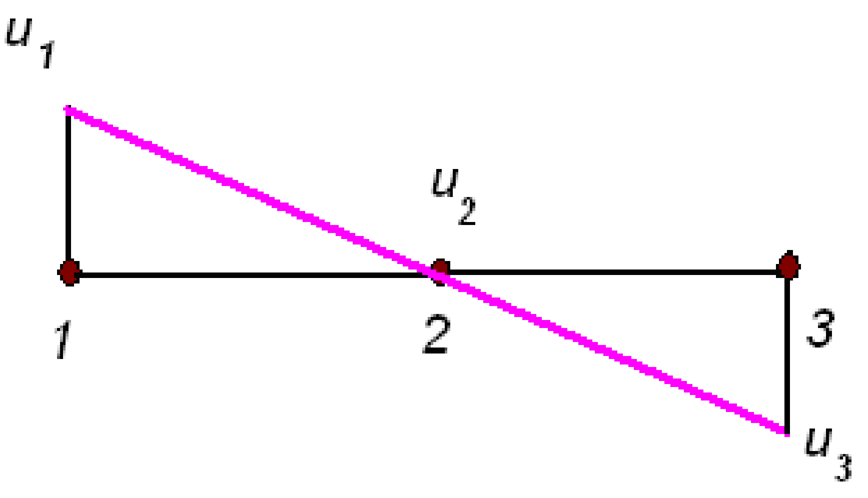
$[K - M\omega^2_n] \{u\} = 0$

after substituting for ω_n as ω_{n_1} or ω_{n_2}

For $\omega_n = \omega_{n_1}$, we get

$$\{u\} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$

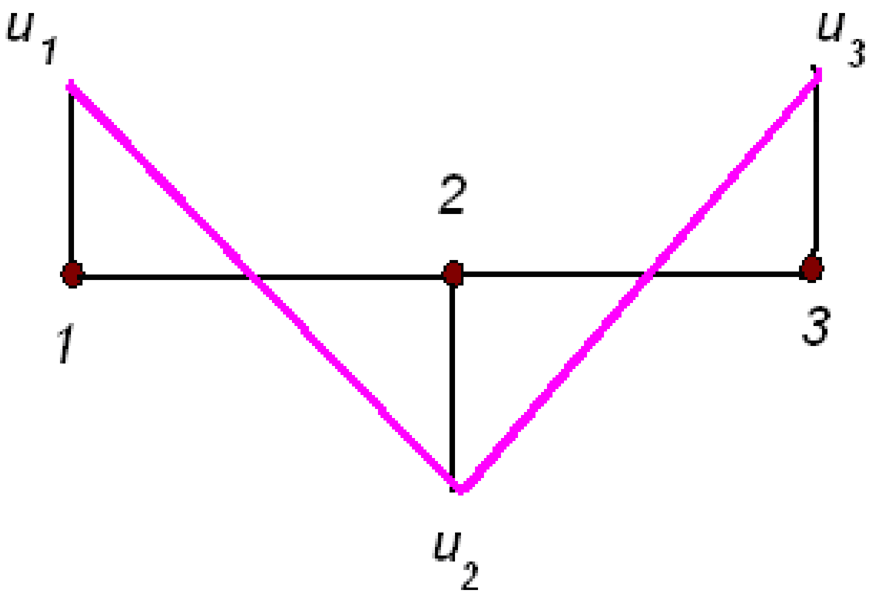
The mode shape is given by



For $\omega_n = \omega_{n_2}$ we get

$$\{u\} = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

The mode shape is given by



LAGRANGIAN INTERPOLATION FUNCTIONS

The Lagrange interpolation polynomials associated with node \`í' of an n^{th} order element is given by,

$$L_i(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_1)(x_i-x_2)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}$$

$$\text{or} \quad L_k(x) = \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}$$

It is seen that $L_k(x)$ is an n^{th} degree polynomial given by the product of n linear factors. It can also be seen that if $x = x_k$, the numerator becomes equal to the denominator and $L_k(x)$ will have a value unity.

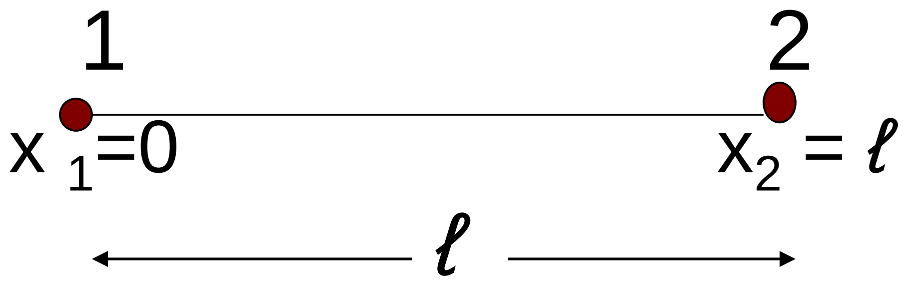
On the other hand, if $x = x_i$ and $í \neq k$ the numerator & hence $L_k(x)$ will become Zero,

ie., $L_k(x_j) = \begin{cases} 1 & \text{if } í = k \\ 0 & \text{if } í \neq k \end{cases}$

Where x_j denotes the x co-ordinate of the i^{th} node in the element.

Linear Element:

We shall derive the shape functions for a two noded linear element using Lagrangian polynomials.



$$L_1(x) = \frac{x - x_2}{x_1 - x_2}$$

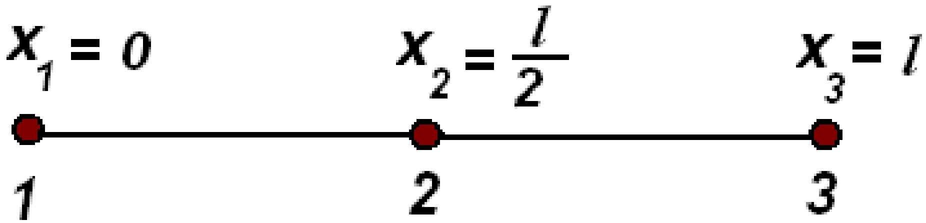
Substituting $x_1 = 0$ & $x_2 = \ell$ we get

$$L_1(x) = \frac{x - \ell}{0 - \ell} = 1 - \frac{x}{\ell}$$

$$L_2(x) = \frac{x - x_1}{x_2 - x_1} = \frac{x}{\ell}$$

which are the same as that obtained by inverting the generalized co-efficient matrix.

Quadratic Element:



$$L_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)\,(x_1-x_3)} = \frac{(x-\ell/2)\;(x-\ell)}{(-\ell/2)\;(-\ell)}$$

$$= \frac{x^2 - x\ell - x\ell/2 + \ell^2/2}{\ell^2/2}$$

$$= \frac{2x^2}{l^2} - \frac{3x}{l} + 1$$

$$L_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(x-0)(x-\ell)}{(\ell/2-0)(\ell/2-\ell)}$$

$$= \frac{4x}{l} - \frac{4x^2}{l^2}$$

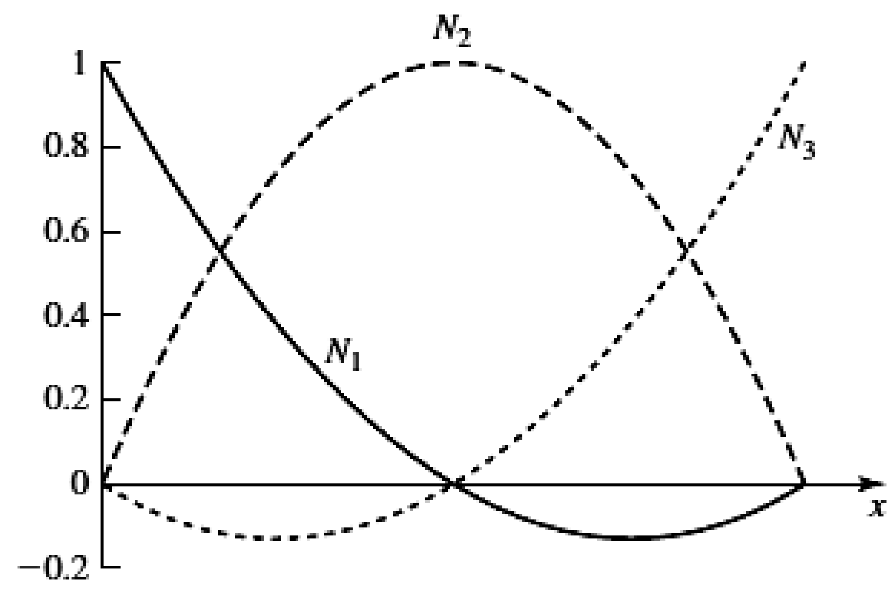
$$L_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(x-0)(x-\ell/2)}{(\ell-0)(\ell-\ell/2)}$$

$$= \frac{2x^2}{l^2} - \frac{x}{l}$$

where $$N1(x) = \frac{2x^2}{l^2} - \frac{3x}{l} + 1$$

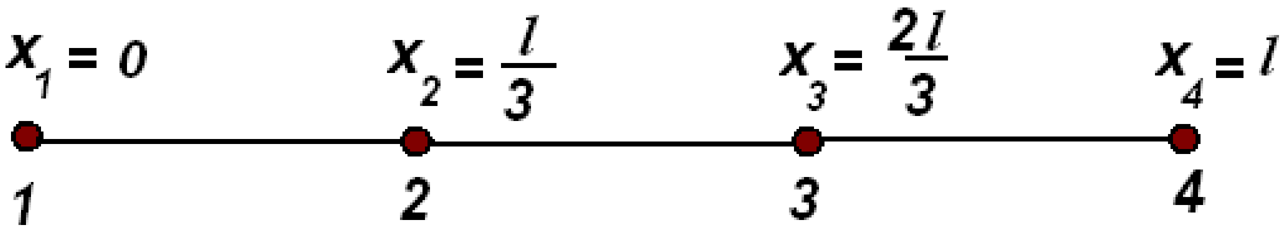
$$N2(x) = \frac{4x}{l} - \frac{4x^2}{l^2}$$

$$N3(x) = \frac{2x^2}{l^2} - \frac{x}{l}$$



Spatial variation of interpolation functions for a three-node line element.

Cubic Element:



$$L_1(x) = \frac{(x-x_2)\,(x-x_3)\,(x-x_4)}{(x_1-x_2)\,(x_1-x_3)\,(x_1-x_4)}$$

$$= \quad (1-3x/\ell)\,(1-3x/2\ell)\,(1-x/\ell)$$

$$L_2(x) = \frac{(x-x_1)\,(x-x_3)\,(x-x_4)}{(x_2-x_1)\,(x_2-x_3)\,(x_2-x_4)}$$

$$= \quad 9x/\ell \;\; (1-3x/2\ell)\,(1-x/\ell)$$

$$L_3(x) = \frac{(x-x_1)\,(x-x_2)\,(x-x_4)}{(x_3-x_1)\,(x_3-x_2)\,(x_3-x_4)}$$

$$= \; -\, 9/2 \;\; x/\ell \;\;\; (1-3x/\ell)\,(1-x/\ell)$$

$$L_4(x) = \frac{(x-x_1)\,(x-x_2)\,(x-x_3)}{(x_4-x_1)\,(x_4-x_2)\,(x_4-x_3)}$$

$$= \quad x/\ell \quad (1-3x/\ell)\,(1-3x/2\ell)$$

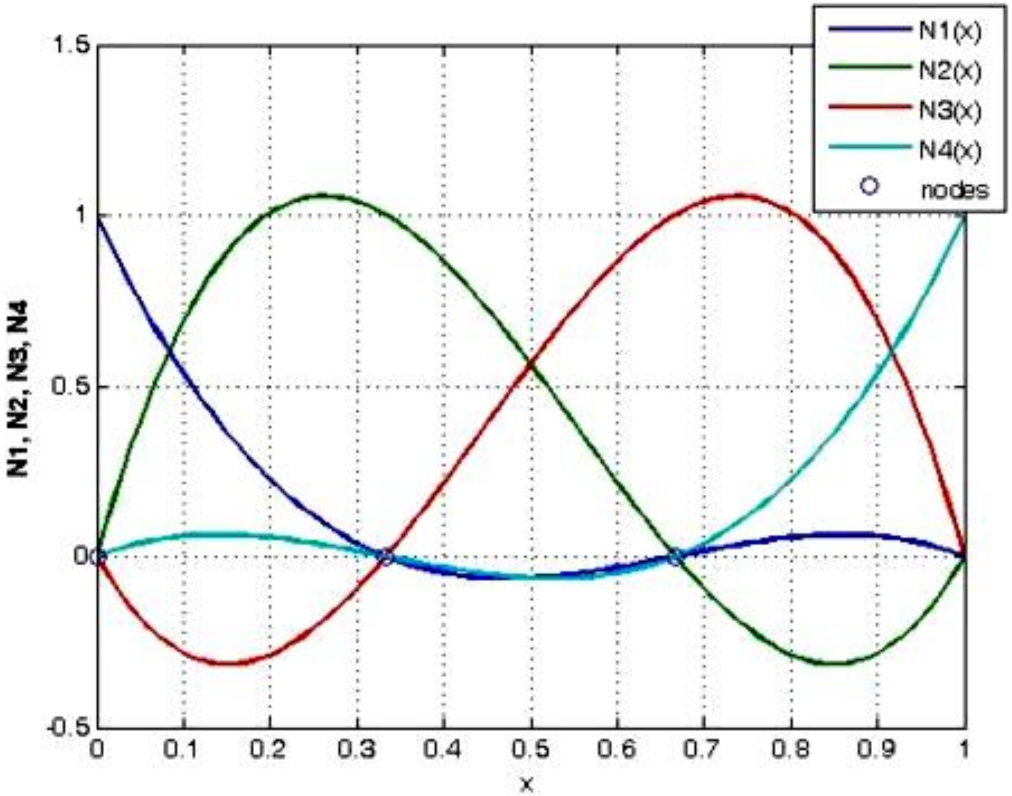
Thus the Lagrangian Polynomials provide us with a quick and easy method of deriving the Shape Functions. It will later be used to derive the shape functions for ID and **2D rectangular** elements using **Natural Co-ordinates**.

$$N_1(x) = (1-3x/\ell)\,(1-3x/2\ell)\,(1-x/\ell)$$

$$N_2(x) = 9x/\ell\ \ (1-3x/2\ell)\,(1-x/\ell)$$

$$N_3(x) = -\,9/2\ \ x/\ell\ \ (1-3x/\ell)\,(1-x/\ell)$$

$$N_4(x) = x/\ell\ \ (1-3x/\ell)\,(1-3x/2\ell)$$

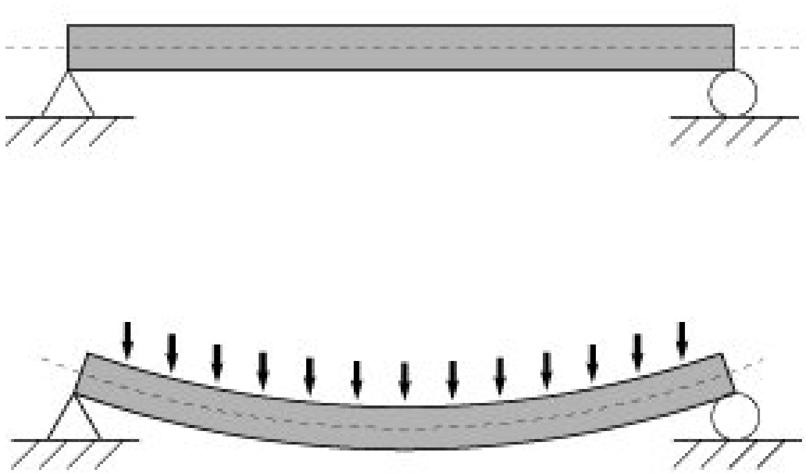


$$\frac{d}{dx}[\mathrm{EA(x)}\ \frac{\mathrm{du}}{\mathrm{dx}}] + \gamma\mathrm{A(x)} = 0$$

$$\frac{d}{dx}\left[-KA\frac{dT}{dx}\right] + hp(T - T_\infty) = 0$$

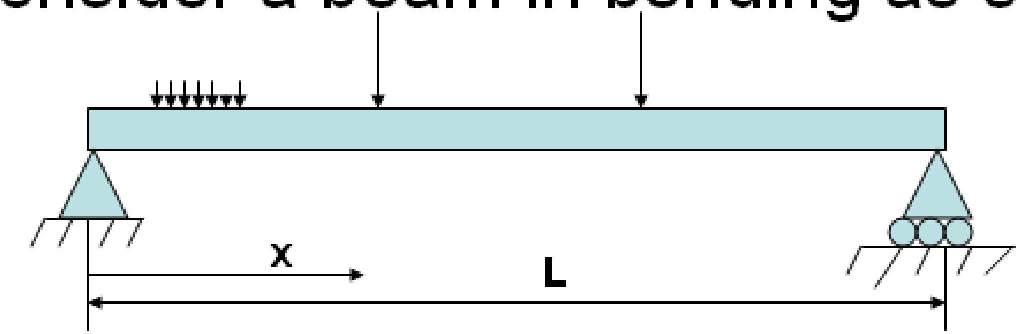
$$\frac{d}{dx}\left[EA\frac{du}{dx}\right] + \rho A(x)u\,\omega_n{}^2 = 0$$

BEAM ELEMENTS

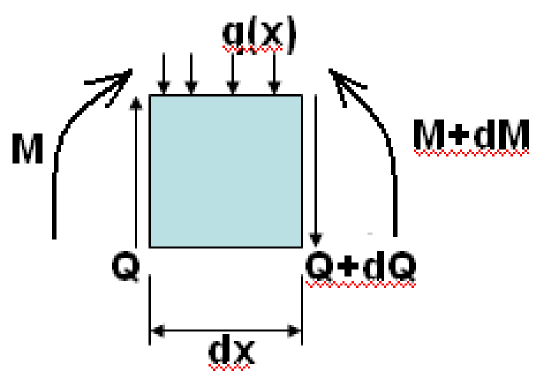


Beams in Bending

- Consider a beam in bending as shown:



Considering an elemental length of the beam



Beam in Bending-Continued

- Considering the equilibrium of vertical forces and moments, we have the governing equation:

$$\frac{dQ}{dx} + q(x) = 0$$

$$\frac{dM}{dx} = Q; \qquad \frac{d^2M}{dx^2} + q(x) = 0$$

$$M = -EI\,\frac{d^2w}{dx^2} \quad and \quad finally$$

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2w}{dx^2} \right) - q(x) = 0$$

Governing Differential Equation

$$EI\frac{d^4 w(x)}{dx^4} = q(x); \qquad q \text{ is the distributed loading}$$

Boundary conditions could involve specification of any of the following variables

$$w \quad = \text{transverse displacement}$$

$$\theta = \frac{dw}{dx} \quad = \text{Slope}$$

$$M = \quad EI\frac{d^2 w}{dx^2} \quad = \text{Moment}$$

$$Q = EI\frac{d^3 w}{dx^3} \quad = \text{Shearforce}$$

$$Boundary\ conditions$$

$$\left.\begin{aligned} w &= transverse\ displacement \\ \theta = \frac{dw}{dx} &= Slope \end{aligned}\right\}$$

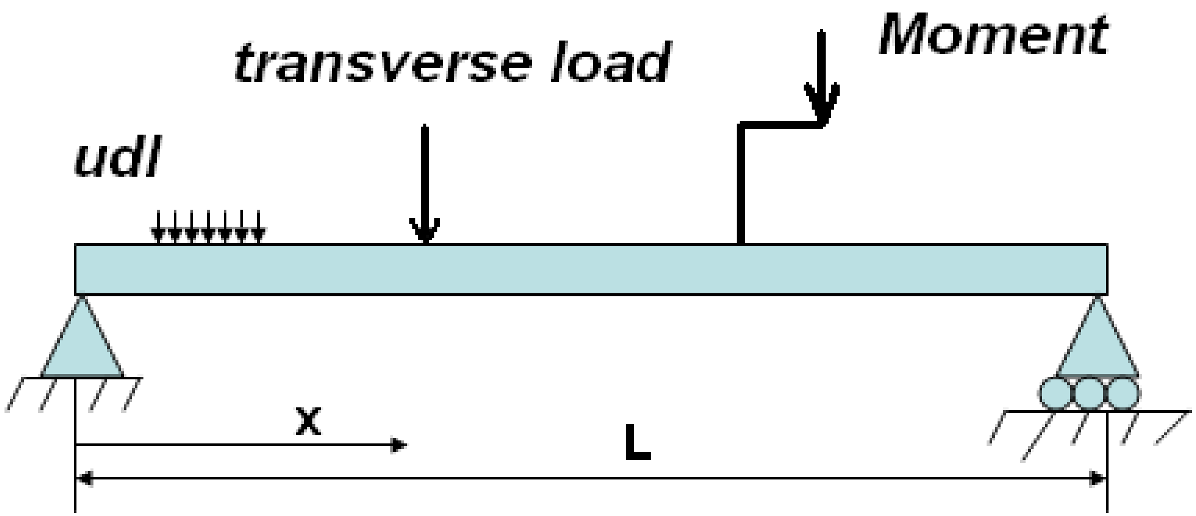
Primary variables

$$\left.\begin{aligned} M = EI\frac{d^2w}{dx^2} &= Moment \\ Q = EI\frac{d^3w}{dx^3} &= Shearforce \end{aligned}\right\}$$

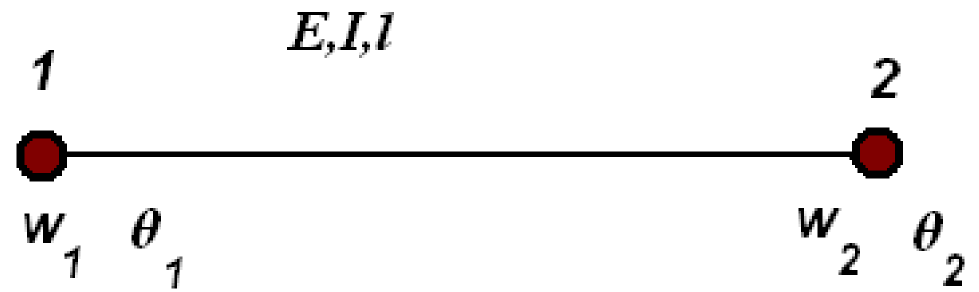
Secondary variables

Possible loads

Distributed load (uniform or non-uniform), Transverse loads, Transverse moments or combination loading in transverse direction



Shape functions for beam element



Sign conventions

$$w\ (x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \Rightarrow (1)$$

$$w\ (x) = < 1 \quad x \quad x^2 \quad x^3 > \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \Rightarrow (2)$$

$$\theta\ (x) = a_1 + 2a_2 x \ + 3a_3 x^2 \Rightarrow (3)$$

$$\theta\ (x) = < 0 \quad x \quad 2x \quad 3x^2 > \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \Rightarrow (4)$$

At x=0 w=w_1 and θ= θ_1
At x=*l* w=w_2 and θ= θ_2

$$\text{at } x = 0 \qquad w_1 = a_0 + a_1 0 + a_2 0 + a_3 0$$

$$\theta_1 = 0 + a_1 + 2a_2 0 + 3a_3 0$$

$$x = l \qquad \mathrm{w}_2 = a_0 + a_1 l + a_2 l^2 + a_3 l^3$$

$$\theta_2 = 0 + a_1 + 2a_2 l + 3a_3 l^2$$

$$\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix}^{-1} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$w(x) = \langle 1 \quad x \quad x^2 \quad x^3 \rangle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix}^{-1} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

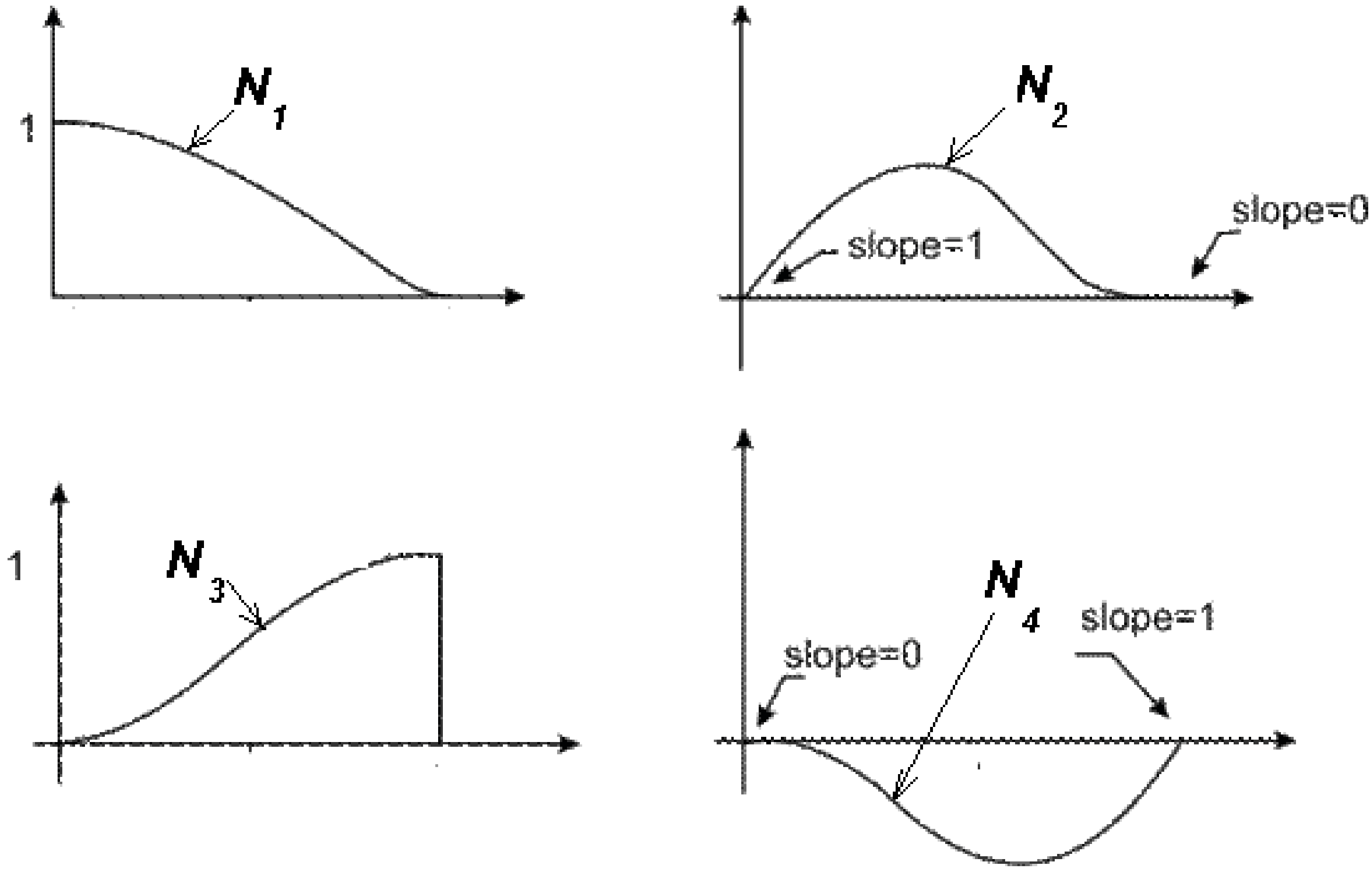
$$w(x) = \langle N_1 \quad N_2 \quad N_3 \quad N_4 \rangle \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$N_1 = 1 - \left(\frac{3x^2}{l^2} \right) + \left(\frac{2x^3}{l^3} \right)$$

$$N_2 = x - \left(\frac{2x^2}{l} \right) + \left(\frac{x^3}{l^2} \right)$$

$$N_3 = \left(\frac{3x^2}{l^2} \right) - \left(\frac{2x^3}{l^3} \right)$$

$$N_4 = -\left(\frac{x^2}{l} \right) + \left(\frac{x^3}{l^2} \right)$$



N_1 & N_2 associated with displacement

N_2 & N_4 associated with slopes

Ritz Weak Formulation

$$\int_0^l \left[EI \frac{d^4 w(x)}{dx^4} - q(x) \right] v(x) dx = 0 \quad v(x) = \text{is the weighting function}$$

$$\int_0^l EI \frac{d^4 w(x)}{dx^4} v(x) dx - \int_0^l q(x) v(x) dx = 0$$

Integration by parts,

$$u = v(x); \qquad dv = EI \frac{d^4 w(x)}{dx^4} \qquad v = EI \frac{d^3 w(x)}{dx^3}$$

$$\left[v(x) EI \frac{d^3 w}{dx^3} \right]_0^l - \int EI \frac{d^3 w}{dx^3} \frac{dv}{dx} dx - \int q(x) v(x) dx = 0$$

Now u = $\dfrac{dv}{dx}$, and du = $\dfrac{d^2 v}{dx^2}$

dv = $EI\dfrac{d^3 w}{dx^3}$, and v = $EI\dfrac{d^2 w}{dx^2}$

$$\left[v(x)EI\frac{d^3 w}{dx^3}\right]_0^l - \left[\frac{dv}{dx}EI\frac{d^2 w}{dx^2}\right]_0^l + \int EI\frac{d^2 w}{dx^2}\frac{d^2 v}{dx^2}dx - \int q(x)v(x)dx = 0$$

Rearranging,

$$\int_0^l EI\frac{d^2 w}{dx^2}\frac{d^2 v}{dx^2}dx = \int_0^1 q(x)v(x)dx + \left[\frac{dv}{dx}EI\frac{d^2 w}{dx^2}\right]_0^l - \left[v(x)EI\frac{d^3 w}{dx^3}\right]_0^l$$

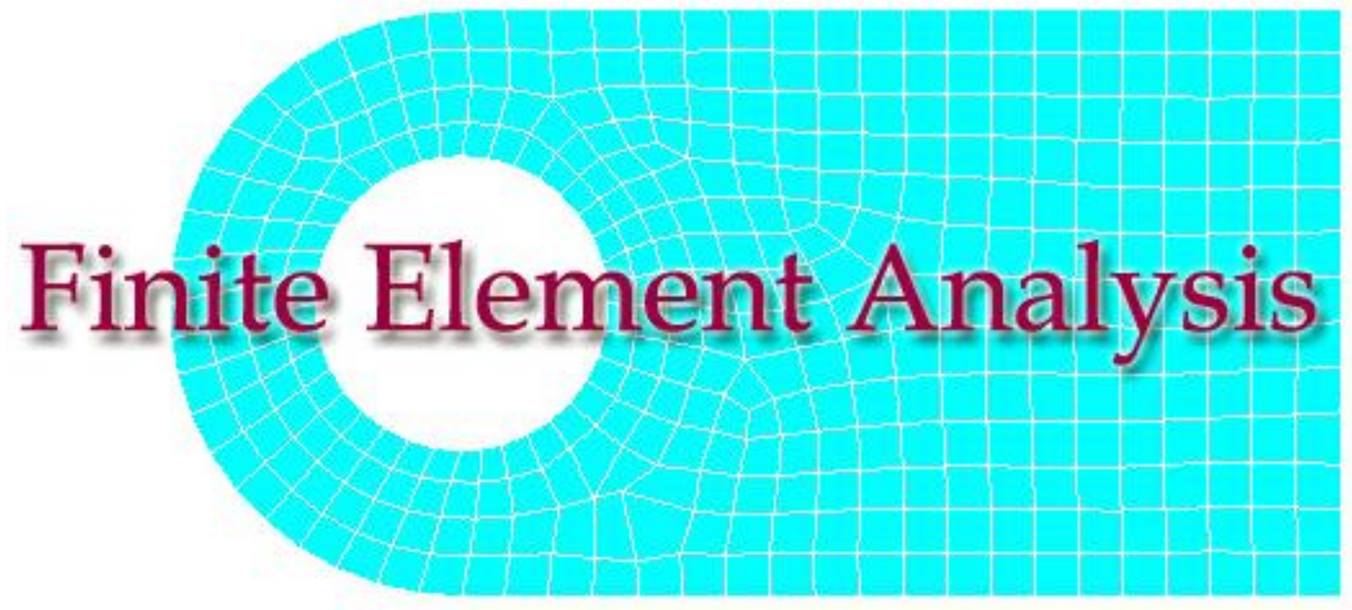
$$\int_0^l EI\,\frac{d^2w}{dx^2}\frac{d^2v}{dx^2}dx = \int_0^1 q(x)v(x)dx + \left[\frac{dv}{dx}EI\,\frac{d^2w}{dx^2}\right]_0^l - \left[v(x)EI\,\frac{d^3w}{dx^3}\right]_0^l$$

$$\int_0^l EI\,\frac{d^2w}{dx^2}\frac{d^2v}{dx^2}dx = \int_0^1 q(x)v(x)dx + \left[\underbrace{\frac{dv}{dx}}_{\textbf{Slope}}\underbrace{EI\,\frac{d^2w}{dx^2}}_{\textbf{Moment}}\right]_0^l - \left[\underbrace{v(x)}_{\textbf{Displacement}}\underbrace{EI\,\frac{d^3w}{dx^3}}_{\textbf{Shear force}}\right]_0^l$$

$$\int_0^l EI \frac{d^2w}{dx^2}\frac{d^2v}{dx^2}dx = \int_0^{\mathrm{l}} q(x)v(x)dx +$$

$$M(l)\theta(l) - M(0)\theta(0) - Q(l)w(l) - Q(0)w(0)$$

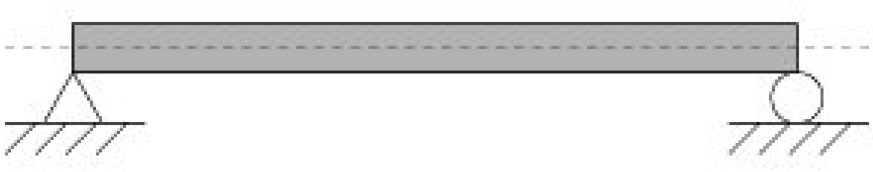
Strain Energy = Work Done by UDL + Work done by moment + Work done by shear force

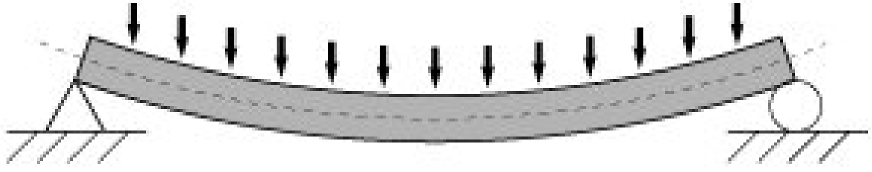


LECTURE 6

1

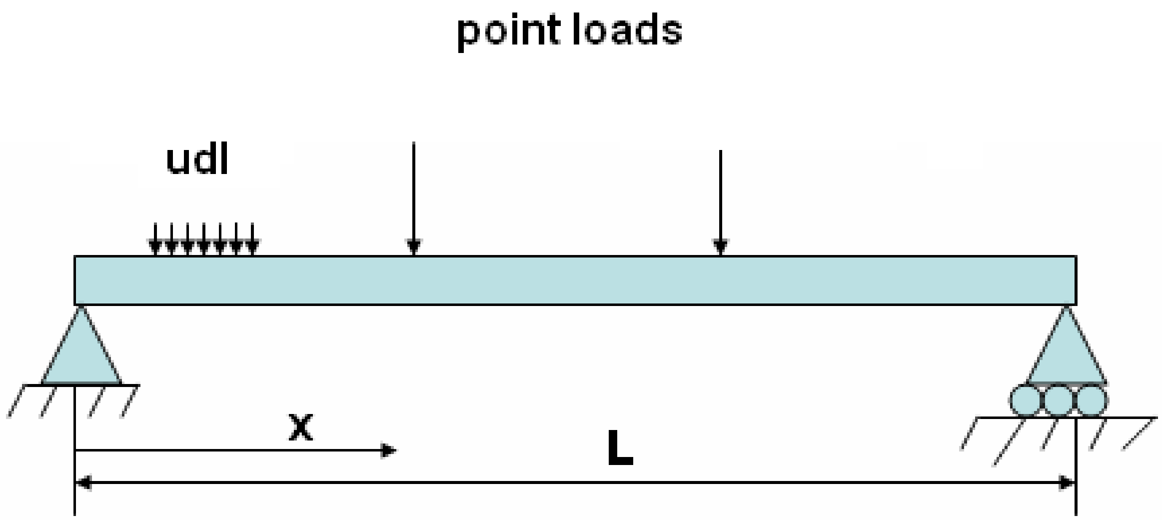
BEAM ELEMENTS



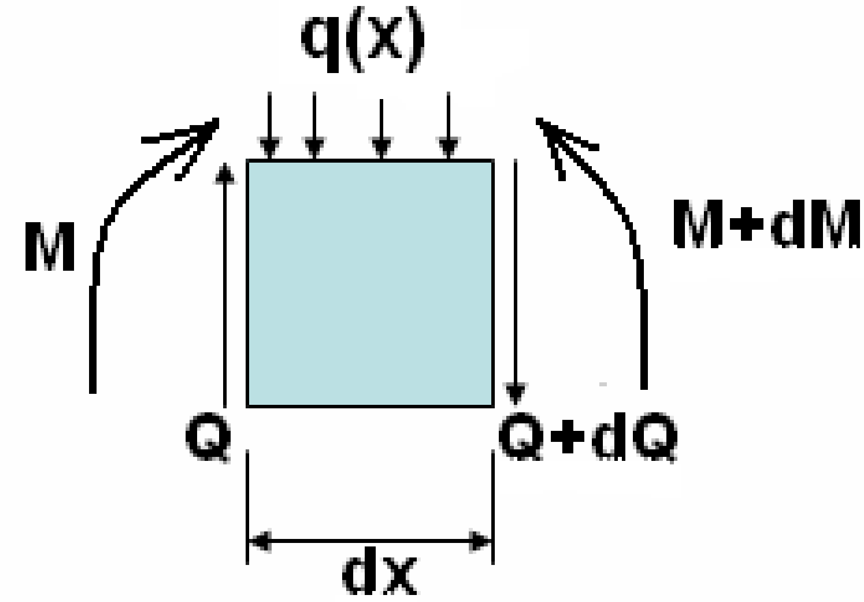


Beams in Bending

- Consider a beam in bending as shown:



Considering an elemental length of the beam



- Considering the equilibrium of vertical forces and moments, we have the governing equation:

$$\frac{dQ}{dx} + q(x) = 0$$

$$\frac{dM}{dx} = Q; \qquad \frac{d^2M}{dx^2} + q(x) = 0$$

$$M = -EI\,\frac{d^2w}{dx^2} \quad and \quad finally$$

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2w}{dx^2} \right) - q(x) = 0$$

Governing Differential Equation

$$EI\frac{d^4 w(x)}{dx^4} = q(x); \qquad q(x) \; is\; the\; distributed\; loading$$

Boundary conditions could involve specification of any of the following variables

$$w \quad = transverse\; displacement$$

$$\theta = \frac{dw}{dx} \quad = Slope$$

$$M = \quad EI\frac{d^2 w}{dx^2} \quad = Moment$$

$$Q = EI\frac{d^3 w}{dx^3} \quad = Shearforce$$

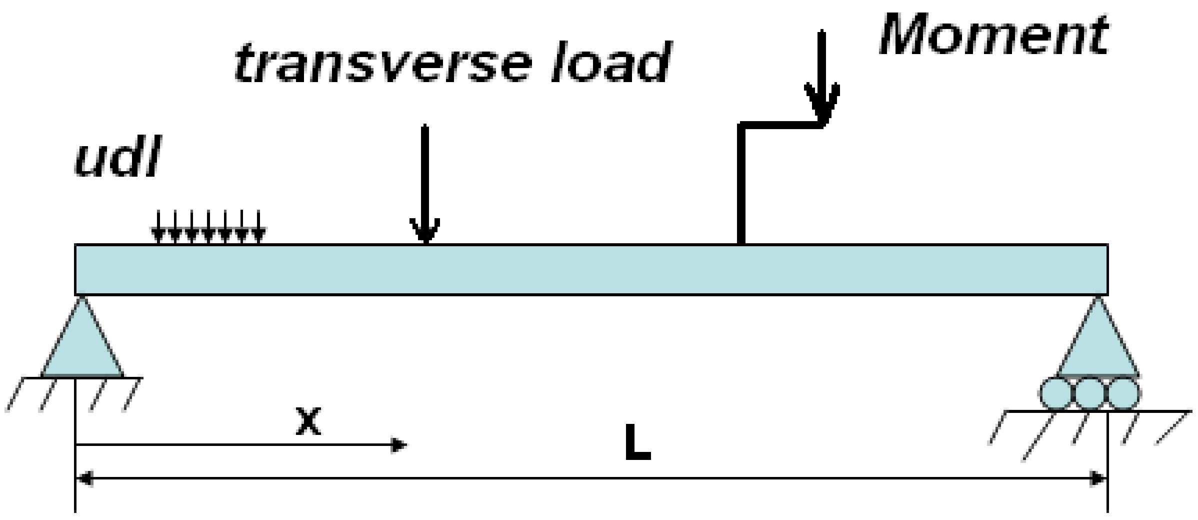
$$Boundary\ conditions$$

$$\left.\begin{array}{l} w \quad = transverse\ displacement \\ \theta = \dfrac{dw}{dx} \quad = Slope \end{array}\right\} \textbf{Primary variables}$$

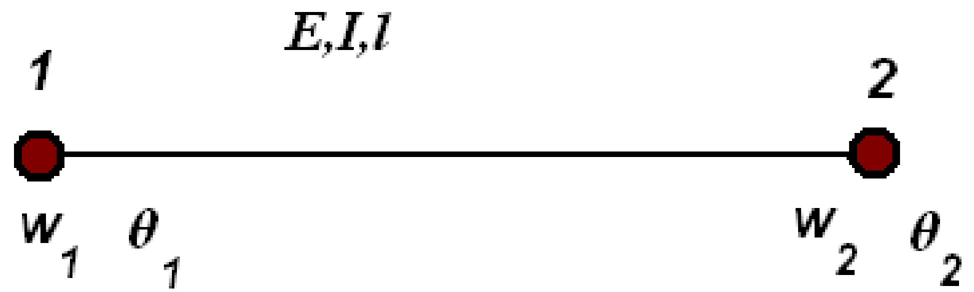
$$\left.\begin{array}{l} M = \quad EI\dfrac{d^2w}{dx^2} \quad = Moment \\ Q = EI\dfrac{d^3w}{dx^3} \quad = Shearforce \end{array}\right\} \textbf{Secondary variables}$$

Possible loads

Distributed load (uniform or non-uniform), Transverse loads, Transverse moments or combination loading in transverse direction



Shape functions for beam element



Sign conventions

$$w(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \Rightarrow (1)$$

$$w(x) = < 1 \quad x \quad x^2 \quad x^3 > \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \Rightarrow (2)$$

$$\theta(x) = a_1 + 2a_2 x + 3a_3 x^2 \Rightarrow (3)$$

$$\theta(x) = < 0 \quad 1 \quad 2x \quad 3x^2 > \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \Rightarrow (4)$$

At x=0 $w=w_1$ and $\theta = \theta_1$

At x=l $w=w_2$ and $\theta = \theta_2$

at $x = 0$ $$w_1 = a_0 + a_1 0 + a_2 0 + a_3 0$$

$$\theta_1 = 0 + a_1 + 2a_2 0 + 3a_3 0$$

$x = l$ $$\mathrm{w}_2 = a_0 + a_1 l + a_2 l^2 + a_3 l^3$$

$$\theta_2 = 0 + a_1 + 2a_2 l + 3a_3 l^2$$

$$\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix}^{-1} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$w(x) = \langle 1 \quad x \quad x^2 \quad x^3 \rangle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix}^{-1} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$w(x) = \langle N_1 \quad N_2 \quad N_3 \quad N_4 \rangle \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

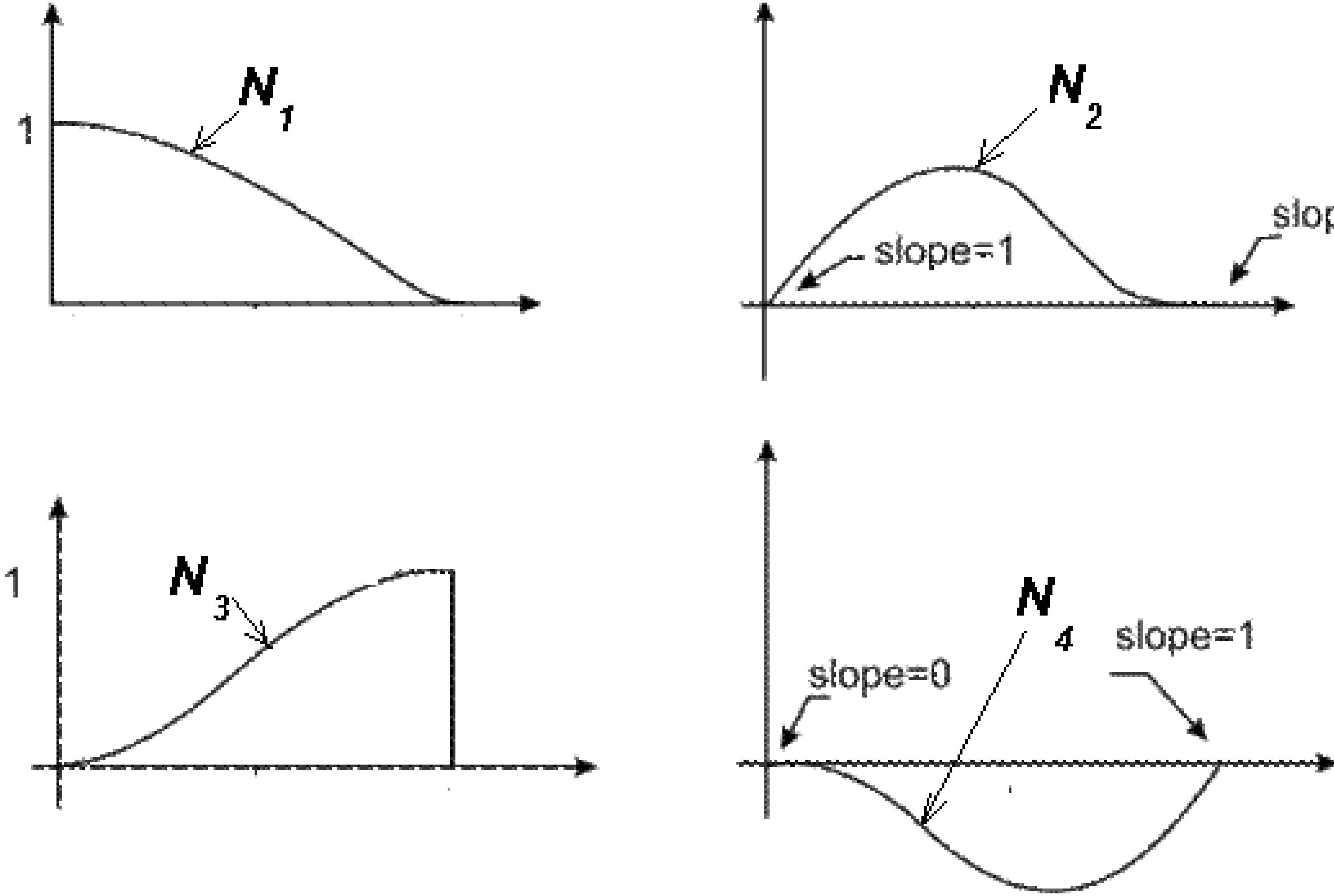
$$N_1 = 1 - \left(\frac{3x^2}{l^2}\right) + \left(\frac{2x^3}{l^3}\right)$$

$$N_2 = x - \left(\frac{2x^2}{l}\right) + \left(\frac{x^3}{l^2}\right)$$

$$N_3 = \left(\frac{3x^2}{l^2}\right) - \left(\frac{2x^3}{l^3}\right)$$

$$N_4 = -\left(\frac{x^2}{l}\right) + \left(\frac{x^3}{l^2}\right)$$

Beams belong to the class of Hermitian polynomials



N_1 & N_3 associated with displacements

N_2 & N_4 associated with slopes

Ritz Weak Formulation

$$\int_0^l \left[EI \frac{d^4 w(x)}{dx^4} - q(x) \right] v(x) dx = 0 \quad v(x) = \text{is the weighting function}$$

$$\int_0^l EI \frac{d^4 w(x)}{dx^4} v(x) dx - \int_0^l q(x) v(x) dx = 0$$

Integration by parts,

$$u = v(x); \qquad dv = EI \frac{d^4 w(x)}{dx^4} \qquad v = EI \frac{d^3 w(x)}{dx^3}$$

$$\left[v(x) EI \frac{d^3 w}{dx^3} \right]_0^l - \int EI \frac{d^3 w}{dx^3} \frac{dv}{dx} dx - \int q(x) v(x) dx = 0$$

Now u = $\dfrac{dv}{dx}$, and du = $\dfrac{d^2v}{dx^2}$

dv = $EI\dfrac{d^3w}{dx^3}$, and v = $EI\dfrac{d^2w}{dx^2}$

$$\left[v(x)EI\frac{d^3w}{dx^3}\right]_0^l-\left[\frac{dv}{dx}EI\frac{d^2w}{dx^2}\right]_0^l+\int EI\frac{d^2w}{dx^2}\frac{d^2v}{dx^2}dx-\int q(x)v(x)dx=0$$

Rearranging,

$$\int_0^l EI\frac{d^2w}{dx^2}\frac{d^2v}{dx^2}dx=\int_0^1 q(x)v(x)dx+\left[\frac{dv}{dx}EI\frac{d^2w}{dx^2}\right]_0^l-\left[v(x)EI\frac{d^3w}{dx^3}\right]_0^l$$

$$\int_0^l EI\,\frac{d^2w}{dx^2}\,\frac{d^2v}{dx^2}dx = \int_0^l q(x)v(x)dx + \left[\frac{dv}{dx}\,EI\,\frac{d^2w}{dx^2}\right]_0^l - \left[v(x)EI\,\frac{d^3w}{dx^3}\right]_0^l$$

$$\int_0^l EI\,\frac{d^2w}{dx^2}\,\frac{d^2v}{dx^2}dx = \int_0^l q(x)v(x)dx + \left[\underbrace{\frac{dv}{dx}}_{\textbf{Slope}}\,\underbrace{EI\,\frac{d^2w}{dx^2}}_{\textbf{Moment}}\right]_0^l - \left[\underbrace{v(x)}_{\textbf{Displacement}}\underbrace{EI\,\frac{d^3w}{dx^3}}_{\textbf{Shear force}}\right]_0^l$$

$$\int_0^l EI \frac{d^2 w}{dx^2}\frac{d^2 v}{dx^2}dx = \int_0^l q(x)v(x)dx +$$

$$M(l)\theta(l) - M(0)\theta(0) - Q(l)w(l) - Q(0)w(0)$$

Strain Energy = Work Done by UDL +
Work done by moment +
Work done by shear force

- From the quadratic functional we see that specification of w and dw/dx= θ constitutes the essential boundary conditions.

- Specification of Q and M constitutes the natural boundary conditions

- Since a quadratic functional exists minimizing it will lead to the equilibrium equations in either the direct form or in the variational (weak) form

Substituting for w(x) and v(x) as given below

$$w(x) = < N_1 \quad N_2 \quad N_3 \quad N_4 > \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

ie

$$w(x) = N_1 w_1 + N_2 \theta_1 + N_3 w_2 + N_4 \theta_2$$

and

$$v(x) = N_1, N_2, N_3, N_4$$

Substituting for the displacement in the weak form of the equation, and taking the weighting functions as the shape functions, we get a system of 4 equations in 4 unknowns.

$$[K]\{u\} = \{f\}$$

$$[K]\{u\}=\{f\}$$

$$where\{u\}=\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$K_{ij}=\int_0^l EI\,\frac{d^2N_i}{dx^2}\frac{d^2N_j}{dx^2}dx$$

$$f_j=\int_0^l q(x)N_j(x)dx$$

Stiffness Matrix for beam element

$$K_{ij} = \int_0^l EI \frac{d^2 N_i}{dx^2} \frac{d^2 N_j}{dx^2} dx$$

$$N_1 = 1 - \left(\frac{3x^2}{l^2} \right) + \left(\frac{2x^3}{l^3} \right)$$

$$N_2 = x - \left(\frac{2x^2}{l} \right) + \left(\frac{x^3}{l^2} \right)$$

$$N_3 = \left(\frac{3x^2}{l^2} \right) - \left(\frac{2x^3}{l^3} \right)$$

$$N_4 = -\left(\frac{x^2}{l} \right) + \left(\frac{x^3}{l^2} \right)$$

$$\frac{dN_1}{dx} = -\frac{6x}{l^2} + \frac{6x^2}{l^3}$$

$$\frac{dN_2}{dx} = 1 - \frac{4x}{l} + \frac{3x^2}{l^2}$$

$$\frac{dN_3}{dx} = \frac{6x}{l^2} - \frac{6x^2}{l^3}$$

$$\frac{dN_4}{dx} = -\frac{2x}{l} + \frac{3x^2}{l^2}$$

$$\frac{d^2N_1}{dx^2} = -\frac{6}{l^2} + \frac{12x}{l^3}$$

$$\frac{d^2N_2}{dx^2} = -\frac{4}{l} + \frac{6x}{l^2}$$

$$\frac{d^2N_3}{dx^2} = \frac{6}{l^2} - \frac{12x}{l^3}$$

$$\frac{d^2N_4}{dx^2} = \frac{2}{l} + \frac{6x}{l^2}$$

$$K_{11} = \int_0^l EI \frac{d^2 N_1}{dx^2} \frac{d^2 N_1}{dx^2} dx$$

$$K_{11} = \int_0^l EI \left(-\frac{6}{l^2} + \frac{12x}{l^3} \right)^2 dx$$

$$= 12 \frac{EI}{l^3}$$

$$K_{12} = \int_0^l EI \frac{d^2 N_1}{dx^2} \frac{d^2 N_2}{dx^2} dx$$

$$= \int_0^l EI \left(-\frac{6}{l^2} + \frac{12x}{l^3} \right) \left(-\frac{4}{l} \quad + \frac{6}{l_2} \right) dx$$

$$= 6 \frac{EI}{l^2} = K_{21}$$

$$K_{13} = \int_0^l EI\left(-\frac{6}{l^2} + \frac{12x}{l^3}\right)\left(\frac{6}{l^2} - \frac{12x}{l^3}\right)dx$$

$$= -12\frac{EI}{l^3} = K_{31}$$

$$K_{14} = \frac{6EI}{l^2} = K_{41}$$

$$K_{22} = \frac{4EI}{l}$$

$$K_{23} = -\frac{6EI}{l^2} = K_{32}$$

$$K_{24} = -\frac{2EI}{l} = K_{42}$$

$$K_{33} = -\frac{12EI}{l^3}$$

$$K_{34} = -\frac{6EI}{l^2} = K_{43}$$

$$K_{44} = -\frac{4EI}{l}$$

$$StiffnessMatrix[K]^e = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

Now the load vector is given by

$$f_j = \int_0^l q(x)N_j(x)dx$$

$$f_1 = \int_0^l q(x)N_1(x)dx = \int_0^l q(x)\left(1-\left(\frac{3x^2}{l^2}\right)+\left(\frac{2x^3}{l^3}\right)\right)dx$$

$$= \frac{ql}{2}$$

$$f_2 = \int_0^l q(x)N_2(x)dx = \int_0^l q(x)\left(x-\left(\frac{2x^2}{l}\right)+\left(\frac{x^3}{l^2}\right)\right)dx$$

$$= \frac{ql^2}{12}$$

$$f_3 = \int_0^l q(x) N_3(x)dx = \int_0^l q(x)\left(\left(\frac{3x^2}{l^2}\right) - \left(\frac{2x^3}{l^3}\right)\right)dx$$

$$= \frac{ql}{2}$$

$$f_4 = \int_0^l q(x) N_4(x)dx = \int_0^l q(x)\left(x - \left(\frac{2x^2}{l}\right) + \left(\frac{x^3}{l^2}\right)\right)dx$$

$$= -\frac{ql^2}{12}$$

Load Vector is given by

$$\{f\}^e = \frac{ql}{2}\begin{Bmatrix} 1 \\ l/6 \\ 1 \\ -l/6 \end{Bmatrix} + \begin{Bmatrix} R \\ 0 \\ 0 \\ M \end{Bmatrix}$$

Hence the element stiffness and load vector for the beam element are given by

$$StiffnessMatrix[K]^e = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

$$\{f\}^e = \frac{ql}{2}\begin{Bmatrix} 1 \\ l/6 \\ 1 \\ -l/6 \end{Bmatrix} + \begin{Bmatrix} R \\ 0 \\ 0 \\ M \end{Bmatrix}$$

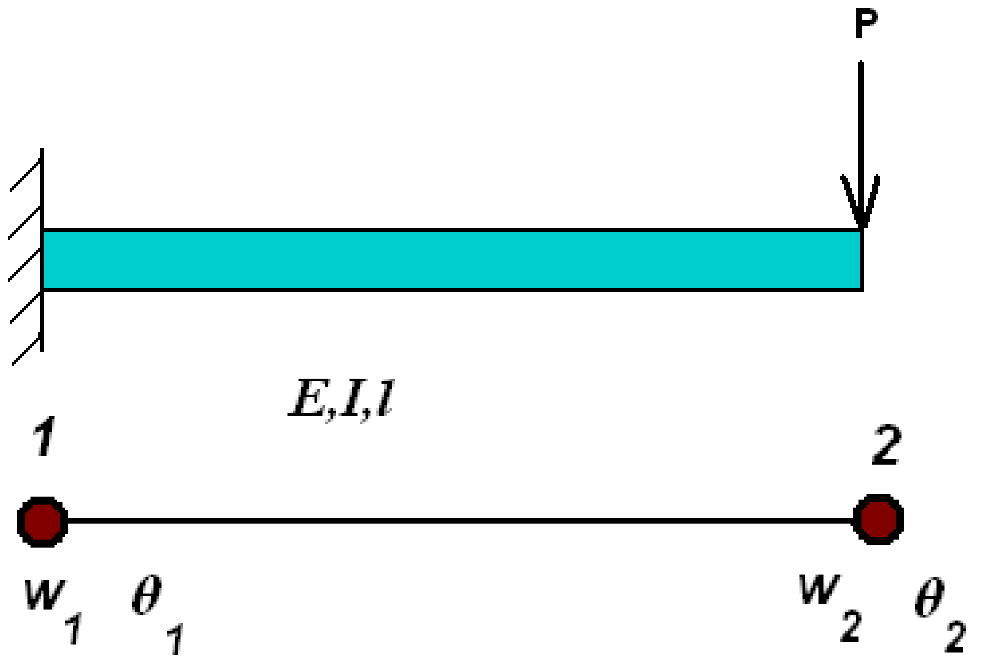
Beam Element

- For a classical beam element,

$$w(x) = \left\langle N_1 \quad N_2 \quad N_3 \quad N_4 \right\rangle \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$\varepsilon_{xx} = \frac{du}{dx} = \frac{d}{dx}\left(z \frac{dw}{dx} \right) = z \frac{d^2 w}{dx^2} = z \left\langle \frac{d^2 N_1}{dx^2} \quad \frac{d^2 N_2}{dx^2} \quad \frac{d^2 N_3}{dx^2} \quad \frac{d^2 N_4}{dx^2} \right\rangle \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

Example 1: Cantilever Beam subjected to point load at the tip



Boundary conditions for this beam are

At $x = 0$ $w_1 = 0$ and $\theta_1 = 0$

At $x = \ell$ $EI \frac{d^3w}{dx^3} = P$ and $EI \frac{d^2w}{dx^2} = M = 0$

The Equilibrium Equation is given by

$$\frac{EI}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} R \\ M \\ -P \\ O \end{Bmatrix}$$

Imposing the essential Boundary conditions we can strike off columns 1 & 2 & Rows 1 & 2 which leaves us with

$$EI\begin{pmatrix} 12 & -6L \\ -6L & 4L^2 \end{pmatrix}\begin{Bmatrix} w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -P \\ O \end{Bmatrix}$$

Which gives the equations.

$$\frac{12\,EI}{L^3} w_2 - \frac{6EI}{L^2} \theta_2 = -P$$

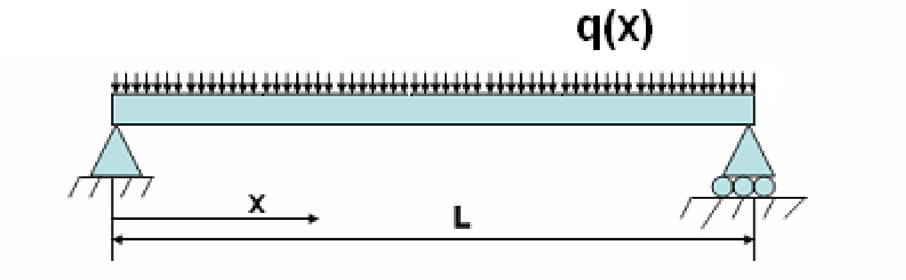
$$-\frac{6EI}{L^2} w_2 + \frac{4EI}{L} \theta_2 = 0$$

Solving for θ_2 & w_2 we get

$$\theta_2 = \frac{PL^2}{2EI}$$

and $w_2 = \dfrac{PL^3}{3EI}$

Example 2: Simply supported beam with uniformly distributed load



The above beam can be idealized by using one element. The entire beam need not be modeled. Instead, taking advantage of symmetry we can model one half of the beam

The boundary conditions in this case are

At $x = 0$ $w_1 = 0$ and $EI \dfrac{d^2w}{dx^2} = 0$

At $x = \ell$ $\theta_2 = 0$ and $EI \dfrac{d^2w}{dx^2} = 0$

The stiffness matrix is given by

$$\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \frac{fl}{2}\begin{Bmatrix} 1 \\ l/6 \\ 1 \\ -l/6 \end{Bmatrix} + \begin{Bmatrix} R \\ 0 \\ 0 \\ M \end{Bmatrix}$$

Where R is the reaction at left end and M is the moment at mid section.

The reduced stiffness matrix after imposing Boundary conditions are given by

$$\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \frac{fl}{2}\begin{Bmatrix} 1 \\ l/6 \\ 1 \\ -l/6 \end{Bmatrix} + \begin{Bmatrix} R \\ 0 \\ 0 \\ M \end{Bmatrix}$$

$$\frac{EI}{l^3}\begin{bmatrix} 4l^2 & -6l \\ -6l & 12 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ w_2 \end{Bmatrix} = \frac{fl}{2}\begin{Bmatrix} l/6 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\frac{4EI\theta_1}{\ell} - \frac{6EI\, w_2}{\ell^2} = \frac{f\ell^2}{12}$$

$$-\frac{6EI\,\theta_1}{\ell^2} + \frac{12EI\, w_2}{\ell^3} = \frac{f\ell}{2}$$

$$\frac{8EI\theta_1}{\ell^2} - \frac{12EI\, w_2}{\ell^3} = \frac{f\ell}{6}$$

$$\theta_1 = \frac{fl^3}{3EI}$$

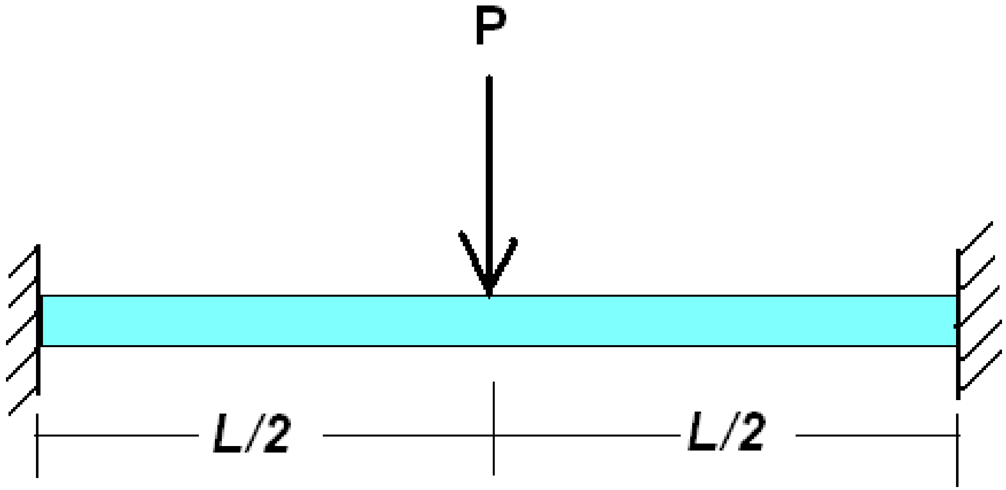
$$w_1 = \frac{5fl^4}{24EI}$$

Substitute ℓ = L/2

We get $$\theta_1 = \frac{fl^3}{24EI}$$

$$w_1 = \frac{5fL^4}{384EI}$$

Example 3: Fixed – Fixed beam with central load



The above beam can be modeled taking advantage of symmetry as a single element

Boundary conditions: at $x = 0$, $w_1 = 0$ & $\theta_1 = 0$

At $x = \ell$, $\theta_2 = 0$ and $EI \dfrac{d^3w}{dx^3} = \dfrac{-P}{2}$

Deleting 1st , 2nd and 4th rows and columns of the stiffness matrix the equilibrium equation is given by

$$\frac{12\,EI}{\ell^3}\,w_2 = -\frac{P}{2}$$

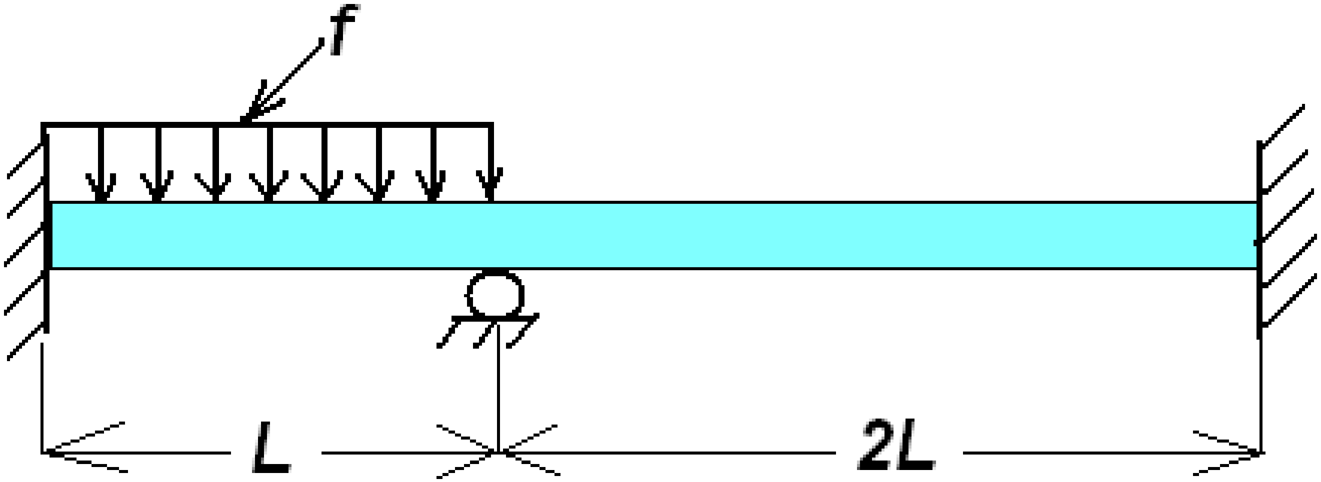
$$\text{or } w_2 = \frac{-P}{2}\;\frac{\ell^3}{12\,EI}$$

$$= \frac{P\ell^3}{24\,EI}\downarrow \quad \text{(down wards)}$$

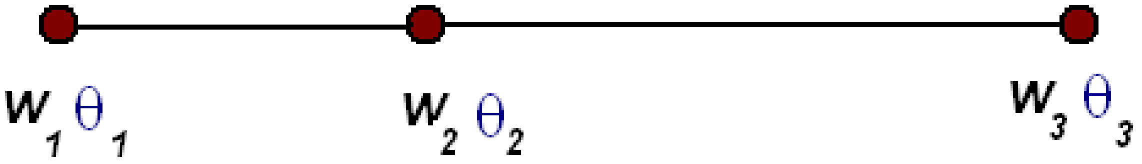
Substituting $\ell = L/2$ we get

$$\boxed{w_2 = \frac{PL^3}{192\,EI}}$$

<u>EXAMPLE 4:</u> The beam shown in fig is fixed at both ends and supported between the ends with a simple support that allows rotation. Compute the rotation and reaction at the supports. Also determine the moments and shear forces.



The given beam can be discretized into two elements as shown below



The stiffness matrix & equations are given by

Element 1

$$\frac{EI}{\ell^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{Bmatrix} = \frac{f\,\ell}{2}\begin{Bmatrix} 1 \\ L/6 \\ 1 \\ -L/6 \end{Bmatrix}$$

Element 2

$$\frac{EI}{[2\ell]^3}\begin{pmatrix} 12 & 6(2L) & -12 & 6(2L) \\ 6(2L) & 4(2L)^2 & -6(2L) & 2(2L)^2 \\ -12 & -6(2L) & 12 & -6(2L) \\ 6(2L) & 2(2L)^2 & -6(2L) & 4(2L)^2 \end{pmatrix}\begin{Bmatrix} w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The global stiffness matrix is a (6 x 6) matrix.

Boundary conditions are

$w_1 = w_2 = w_3 = \theta_1 = \theta_3 = 0$

The global equations now reduces to one equation and one unknown, θ_2 [Remove 1st, 2nd, 3rd, 5th, & 6th rows & columns].

$$\frac{EI}{L^3}\,(4L^2 + 2L^2)\,\theta_2 = \frac{fL^2}{12}$$

or

$$\theta_2 = \frac{fL^3}{72\,EI}$$

Now to compute reactions and moments for each span we utilize the local stiffness matrix for that span. Let the reactions and moments for the span 1-2 be R_1, M_1, R_2 and M_2.

$$\frac{EI}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{fL^3}{72\,EI} \end{Bmatrix} = \frac{f\,\ell}{2}\begin{Bmatrix} 1 \\ L/6 \\ 1 \\ -L/6 \end{Bmatrix} + \begin{Bmatrix} R_1 \\ M_1 \\ R_2 \\ M_2 \end{Bmatrix}$$

Solving we get

$$R_1 = \frac{7fL}{12} \; ; \; M_1 = \frac{fL^2}{9} \; ; \; R^1_2 = \frac{5fL}{12} \; ; \; M_2 = - \frac{WL^2}{36}$$

R_2 represents the reaction at node 2 which is the sum of shear forces at 2nd node of element (1) and that at the 1st node of element (2). Thus $R_2 = R^1_2 + R^2_2$.
The stiffness matrix for element (2) can be used to compute R^2_2, M_2, R_3 and M_3.

$$\frac{EI}{8L^3}\begin{pmatrix} 12 & 12L & -12 & 12L \\ 12L & 16L^2 & -12L & 8L^2 \\ -12 & -12L & 12 & -12L \\ 12L & 8L^2 & -2L & 16L^2 \end{pmatrix} \begin{Bmatrix} 0 \\ \dfrac{fL^3}{72\,EI} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} R^2_2 \\ M_2 \\ R_3 \\ M_3 \end{Bmatrix}$$

Solving we get $R^2_2 = \dfrac{fL}{48}$ $\quad R_3 = \dfrac{-fL}{48}$

$M_2 = \dfrac{fL^2}{36}$ $\quad M_3 = \dfrac{fL^2}{72}$

$R_2 = R^1_2 + R^2_2$

VIBRATION OF BEAMS

The 2 Noded Beam element can be used to determine the natural frequency of transverse vibration. The governing equations for transverse vibration of a beam is given by

$$EI\frac{d^4w}{dx^4} - \rho\frac{d^2w}{dt^2} = 0 \qquad \rightarrow (1)$$

This can be converted to a different form by considering

$w = W\sin\omega_n t$ $\qquad \frac{dw}{dt} = W\omega_n\cos\omega_n t$

$$\therefore \quad \frac{d^2w}{dt^2} = -\omega_n^2 W\sin\omega_n t = -\omega_n^2 w$$

$$\therefore EI \frac{d^4w}{dx^4} + \rho w \omega_n{}^2 = 0$$

The weak form of this eqn. is given by

$$\int_0^l EI \frac{d^2w}{dx^2}\frac{d^2v}{dx^2}dx - \int_0^l \rho A w(x) v(x) dx \omega_n{}^2 = 0$$

Substituting for w(x) and v(x) as given below

$$w(x) =< N_1 \quad N_2 \quad N_3 \quad N_4 > \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$ie$$

$$w(x) = N_1 w_1 + N_2 \theta_1 + N_3 w_2 + N_4 \theta_2$$

$$and$$

$$v(x) = N_1, N_2, N_3, N_4$$

$$K_{ij} = \int_0^l EI \frac{d^2 N_i}{dx^2} \frac{d^2 N_j}{dx^2} dx$$

$$M_{ij} = \int_0^{\mathrm{l}} \rho A N_i N_j dx \quad = 0$$

The elemental matrixes are given by

$$\text{Stiffness Matrix } [K] = \frac{EI}{\ell^3} \begin{pmatrix} 12 & 6L & -12L & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}$$

$$\text{Mass Matrix } [M] = \frac{\rho AL}{420} \begin{pmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L2 & -13L & -3L2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L2 & -22L & 4L2 \end{pmatrix}$$

The Eigen Value problem is given by

$$[K]\{w\} - [M]\,\omega_n^2\,\{w\} = 0$$

$$\text{or} \left([K] - [M]\,\omega_n^{\ 2}\right)\{w\} = 0$$

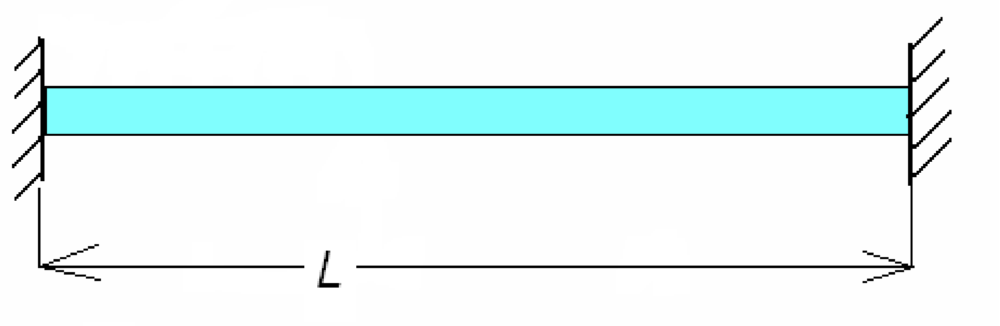
Here {w} gives the eigen vector or the vector that defines the mode shape corresponding to each eigen value ω_n (Natural frequency).

$$\text{Since}\,\{w\} \neq 0 \left|[K] - [M]\right| \omega_n^{\ 2} = 0$$

This equation can be solved for natural frequencies.

Example 1

Natural Frequency of a fixed – fixed Beam



Boundary conditions are $w_1 = \theta_1 = \theta_2 = 0$. Therefore the eigen value equation reduces to the following.

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

$$[M]^e = \frac{\rho A \ell}{420}\begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 \\ 54 & 13\ell & 156 & -22\ell \\ -13\ell & -3\ell^2 & -22\ell & 156 \end{bmatrix}$$

$$\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} - \frac{\rho A \ell}{420}\begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 \\ 54 & 13\ell & 156 & -22\ell \\ -13\ell & -3\ell^2 & -22\ell & 156 \end{bmatrix}\omega_n{}^2 = 0$$

$$\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & 6l & 4l^2 \end{bmatrix} - \frac{\rho A \ell}{420}\begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 \\ 54 & 13\ell & 156 & -22\ell \\ -13\ell & -3\ell^2 & -22\ell & 156 \end{bmatrix}\omega_n{}^2 = 0$$

$$12\frac{EI}{l^3} - \frac{156\rho A\ell}{420}\omega_n{}^2 = 0$$

Dividing throughout by $12EI/\ell^3$ and solving for ω_n we get

$$\omega_n = \frac{5.68}{l^2}\sqrt{\frac{EI}{A\rho}}$$

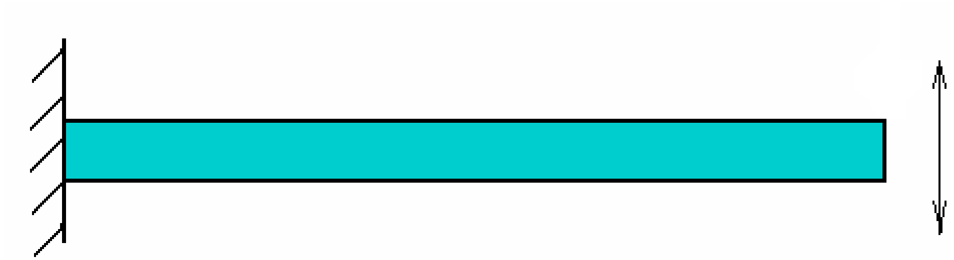
Substitute $L = \ell/2$

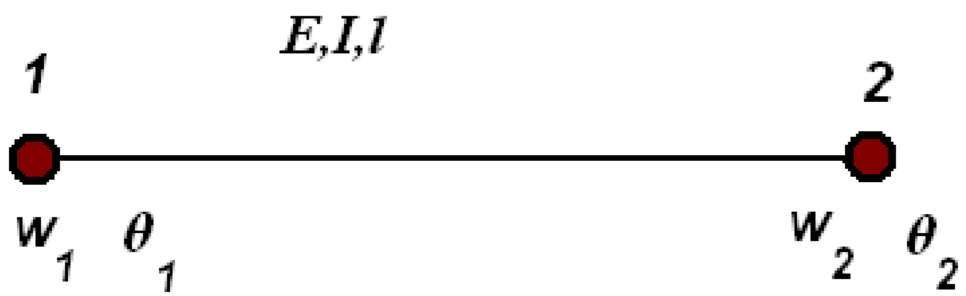
$$\omega_n = \frac{22.735}{L^2}\sqrt{\frac{EI}{A\rho}}$$

Note:-

- In such vibration problems if we require first two natural frequencies then we shall have to discretize the beam into two elements, which will give 2 positive roots.
- The lower frequency represents the first (fundamental) natural frequency and the higher the second natural frequency.
- Substituting the natural frequencies we can obtain the nodal displacements which represents the mode shape.

Example 2: Natural frequency of cantilever Beam





Boundary conditions for this beam are

At $x = 0$ $w_1 = 0$ and $\theta_1 = 0$

At $x = \ell$ $EI \dfrac{d^3w}{dx^3} = 0$ and $EI \dfrac{d^2w}{dx^2} = M = 0$

$$\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} - \frac{\rho A\ell}{420}\begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 \\ 54 & 13\ell & 156 & -22\ell \\ -13\ell & -3\ell^2 & -22\ell & 156 \end{bmatrix}\omega_n^{\ 2} = 0$$

Dividing throughout by EI/l^3 and putting

$$\frac{\rho A\ell^4}{420EI} = \lambda$$

$(12 - 156\lambda)(4L^2 - 4L^2\lambda) - (22\ell\lambda - 6L)^2 = 0$

Dividing throughout $4L^2$

$(12 - 156\lambda)(1 - \lambda) - (11\lambda - 3)^2 = 0$

$35\lambda^2 - 102\lambda + 3 = 0$

Solving for the roots of the above equation we get when $\lambda_1 = 0.03$ and $\lambda_2 = 2.88$

when λ_1 = 0.03

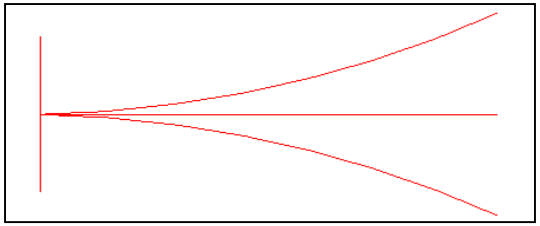
$$\omega_n = \frac{3.55}{l^2}\sqrt{\frac{EI}{A\rho}}$$

When λ_2 =2 .88

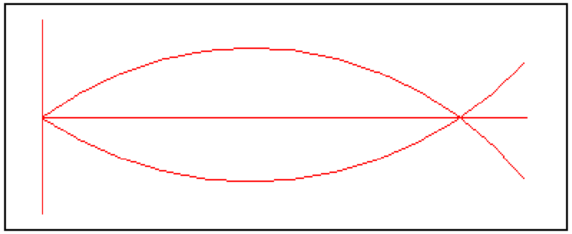
$$\omega_n = \frac{34.78}{l^2}\sqrt{\frac{EI}{A\rho}}$$

Mode Shapes for Cantilever beam

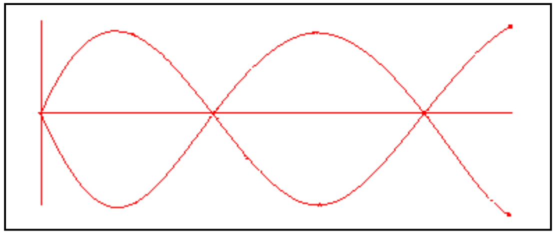
First mode shape



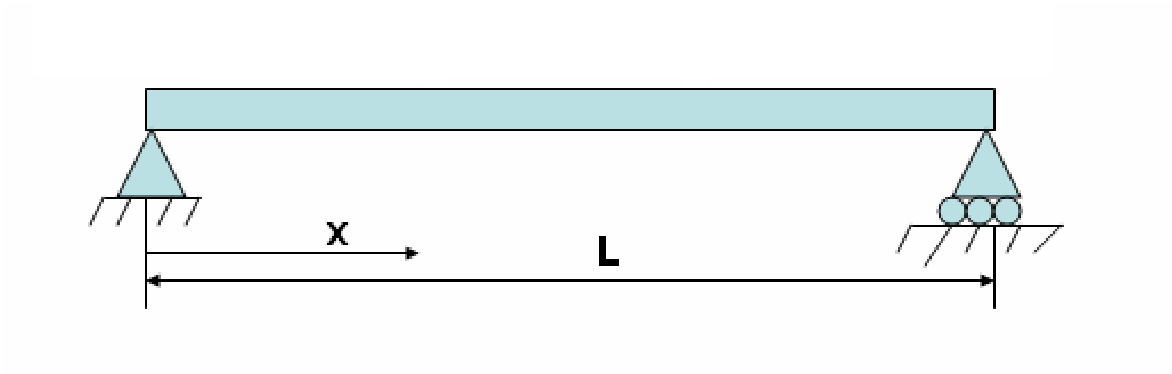
Second mode shape



Third mode shape



Natural frequency of vibration of a simply supported beam:



(2) Boundary Condition : $w_1 = 0$ & $\theta_2 = 0$

$$\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & \boxed{4l^2} & \boxed{-6l} & 2l^2 \\ -12 & \boxed{-6l} & \boxed{12} & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} - \frac{\rho A\ell}{420}\begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ 22\ell & \boxed{4\ell^2} & \boxed{13\ell} & -3\ell^2 \\ 54 & \boxed{13\ell} & \boxed{156} & -22\ell \\ -13\ell & -3\ell^2 & -22\ell & 156 \end{bmatrix}\omega_n^2 = 0$$

$\therefore$ Equilibrium Equation is

$$\frac{EI}{\ell^3}\begin{pmatrix} 4\ell^2 & 2\ell^2 \\ 2\ell^2 & 4\ell^2 \end{pmatrix} - \frac{\rho A\ell\omega_n^2}{420}\begin{pmatrix} 4\ell^2 & -3\ell^2 \\ -3\ell^2 & 4\ell^2 \end{pmatrix} = 0$$

Solving the above we get

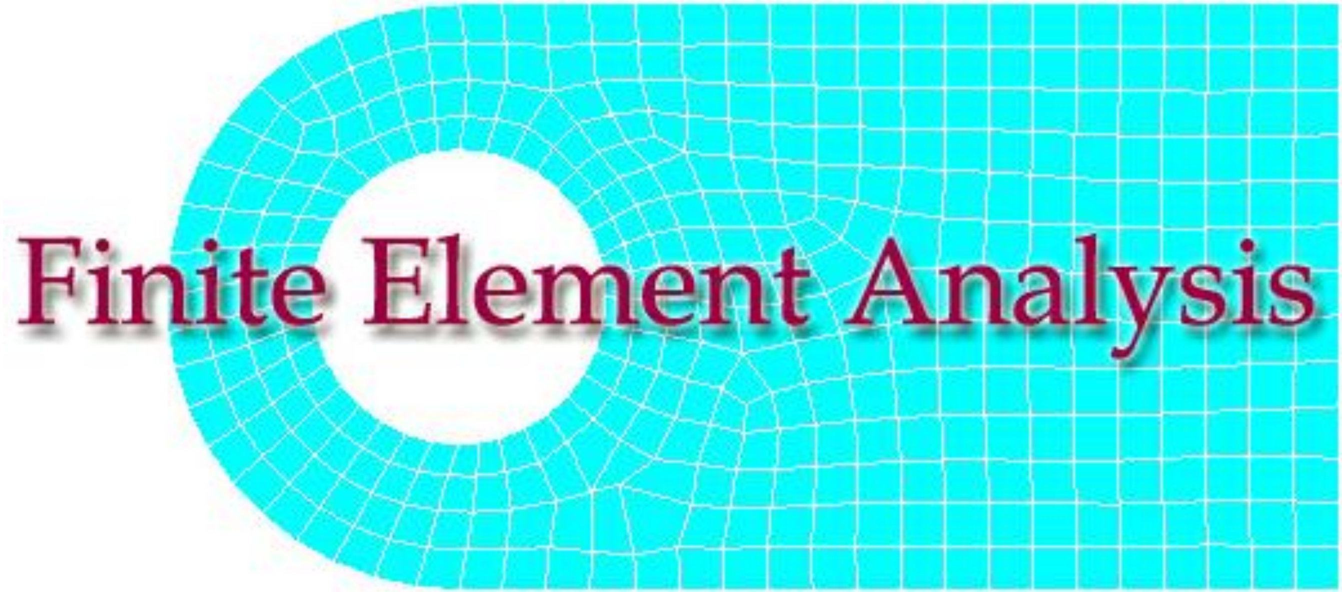
$$\omega_{n_1} = \frac{10.94}{l^2}\sqrt{\frac{EI}{A\rho}}$$

$$\omega_{n2} = \frac{50.12}{l^2}\sqrt{\frac{EI}{A\rho}}$$

$$[K]=\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

$$[M]^e = \frac{\rho A \ell}{420}\begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 \\ 54 & 13\ell & 156 & -22\ell \\ -13\ell & -3\ell^2 & -22\ell & 156 \end{bmatrix}$$

$$\{f\}^e = \frac{ql}{2}\begin{Bmatrix} 1 \\ l/6 \\ 1 \\ -l/6 \end{Bmatrix} + \begin{Bmatrix} R \\ 0 \\ 0 \\ M \end{Bmatrix}$$



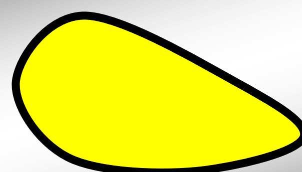
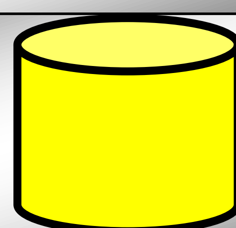
TWO DIMENSIONAL ELEMENTS

LECTURE 7

DIMENSIONALITY

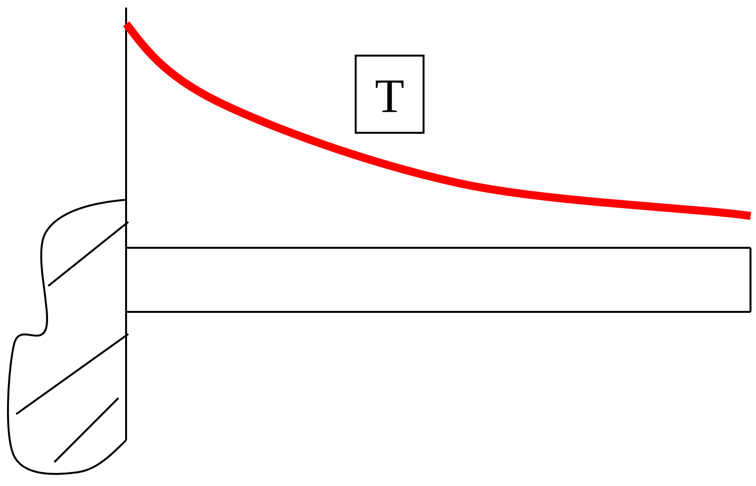
Physical problems can be classified into

(i) I dimensional
(ii) II dimensional
(iii) III dimensional problems.

Domain	Geometry	Boundary
1D	Line	Points
2D	Area 	Curves
3D	Volume 	Area

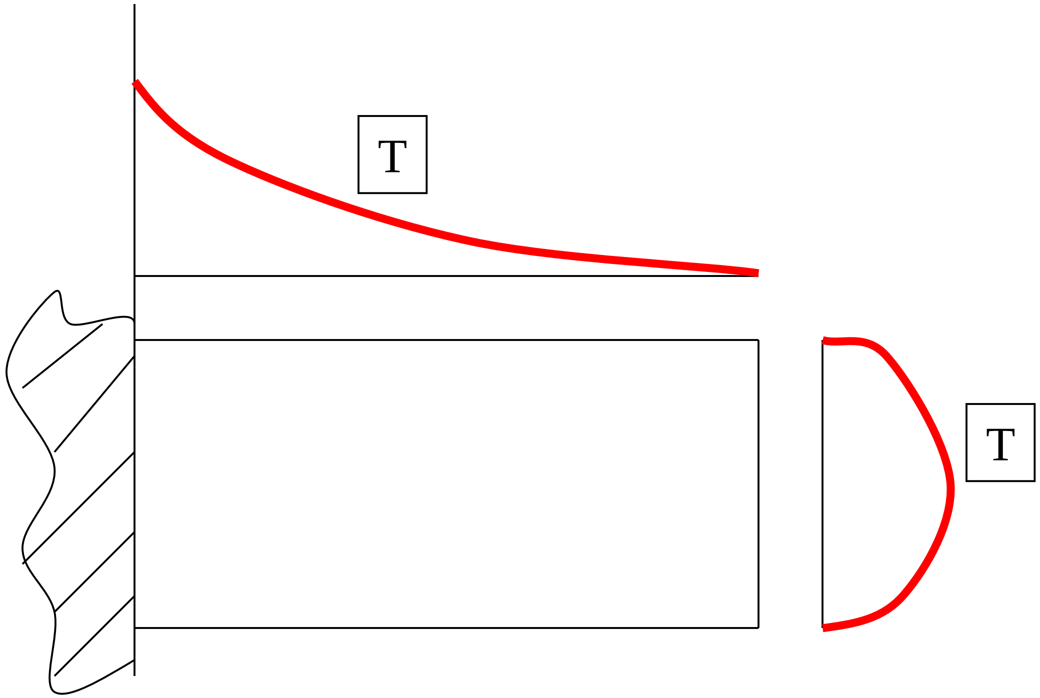
I-D PROBLEMS:-

When the geometry, material properties and field variables such as displacement, temperature, pressure etc can be described in terms of only one spatial co-ordinate we can go in for one-dimensional modeling

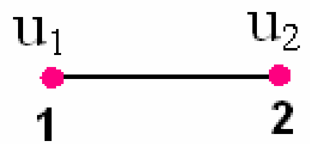


2D PROBLEMS:-

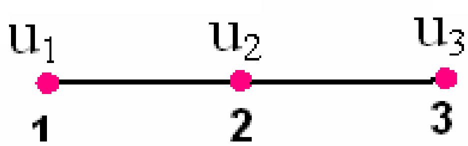
When the geometry and other parameters are described in terms of two independent co-ordinates we go in for two-dimensional modeling.



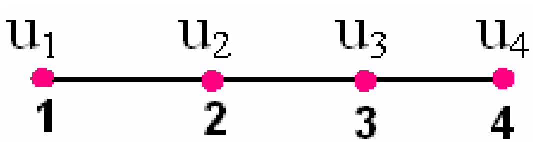
I D elements



2 NODED LINEAR ELEMENT



3 NODED QUADRATIC ELEMENT

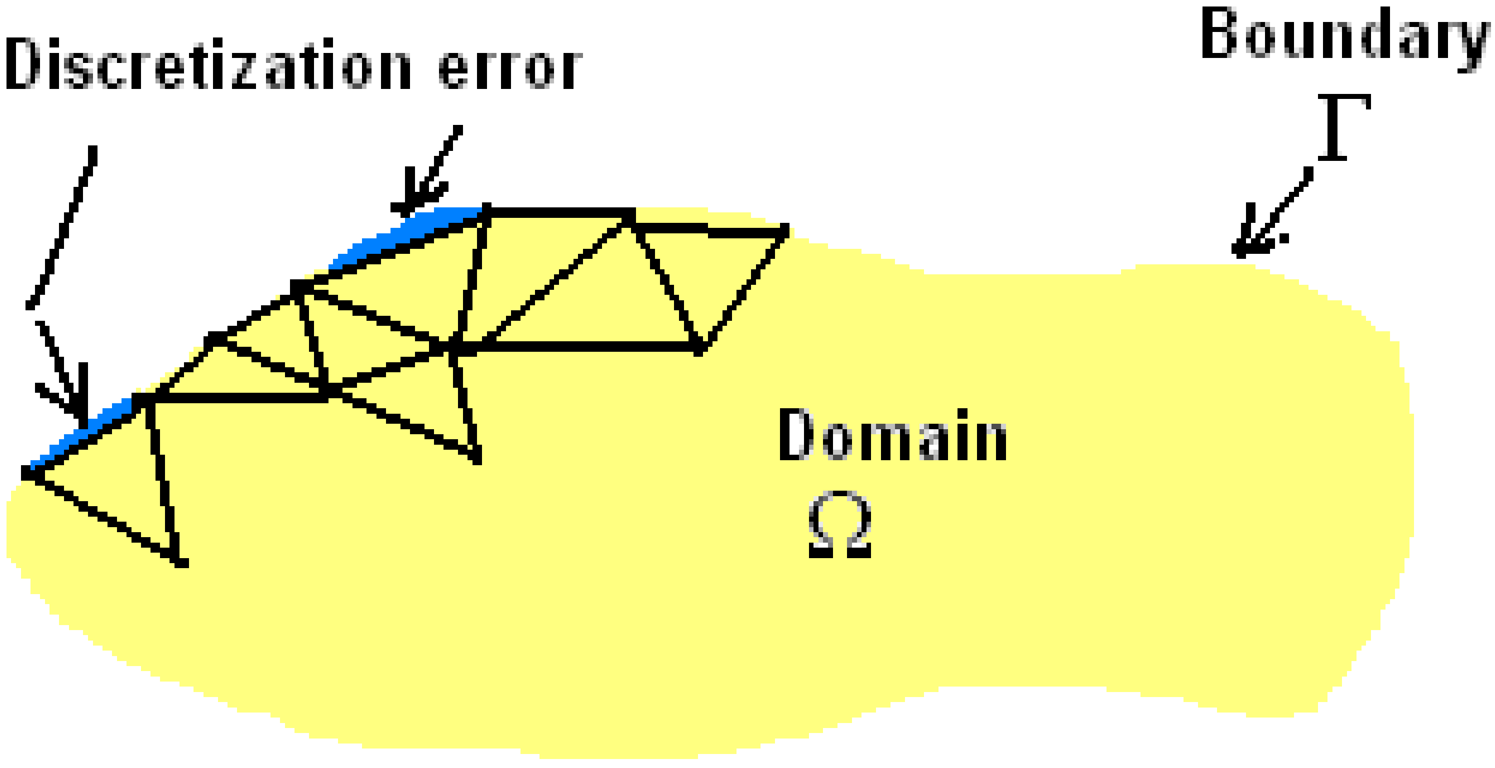


4 NODED CUBIC ELEMENT

Used when field variable varies along the axial dirction

2 NODED BEAM ELEMENT

Used when field variable varies perpendicular to the axis



Two dimensional domain discretised using triangular elements

- 2D problems are described by partial differential equations over geometrically complex regions.

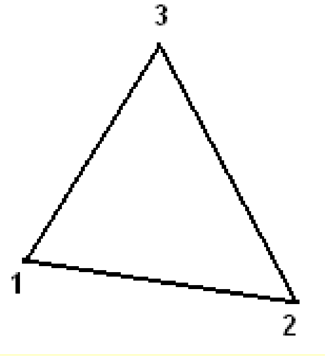
- The boundary of a two dimensional domain is in general a curve i.e. the field variable varies with respect to x & y axes.

- Therefore the finite elements are simple 2D geometric shapes that can be used to approximate a given 2D domain as well as the solution over it.

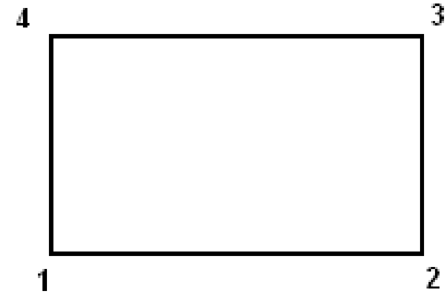
- **Consequently in the Finite Element Analysis of 2D problems we have two approximation errors.**

- **Approximation errors due to approximation of solution over the element.**

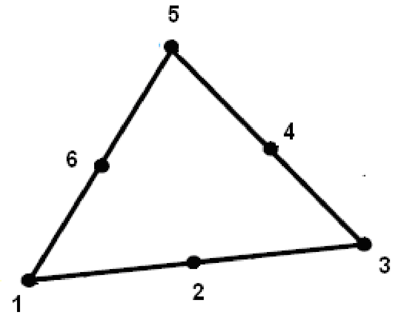
- **Discretisation errors due to the approximation of the domain into finite elements.**



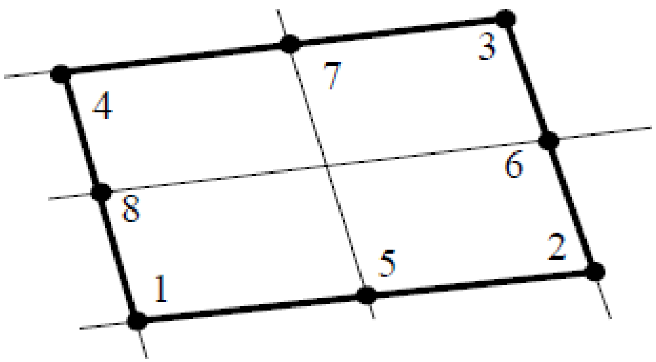
Constant strain triangular element



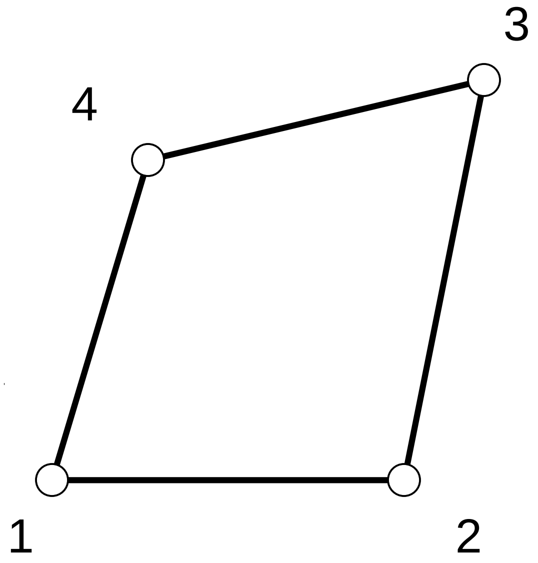
Bilinear Rectangular element



Linear strain triangular element



Eight noded quadratic quadrilateral elements



Linear Quadrilateral element

General form of a 2 D second order equation is given as

$$a_{11}\frac{\partial\ \phi}{\partial x^2} + a_{22}\frac{\partial^2\phi}{\partial y^2} + a_{12}\frac{\partial^2\phi}{\partial x\partial y} + a_{21}\frac{\partial^2\phi}{\partial x\partial y} - a_{00}\, f + f(x, y) = 0$$

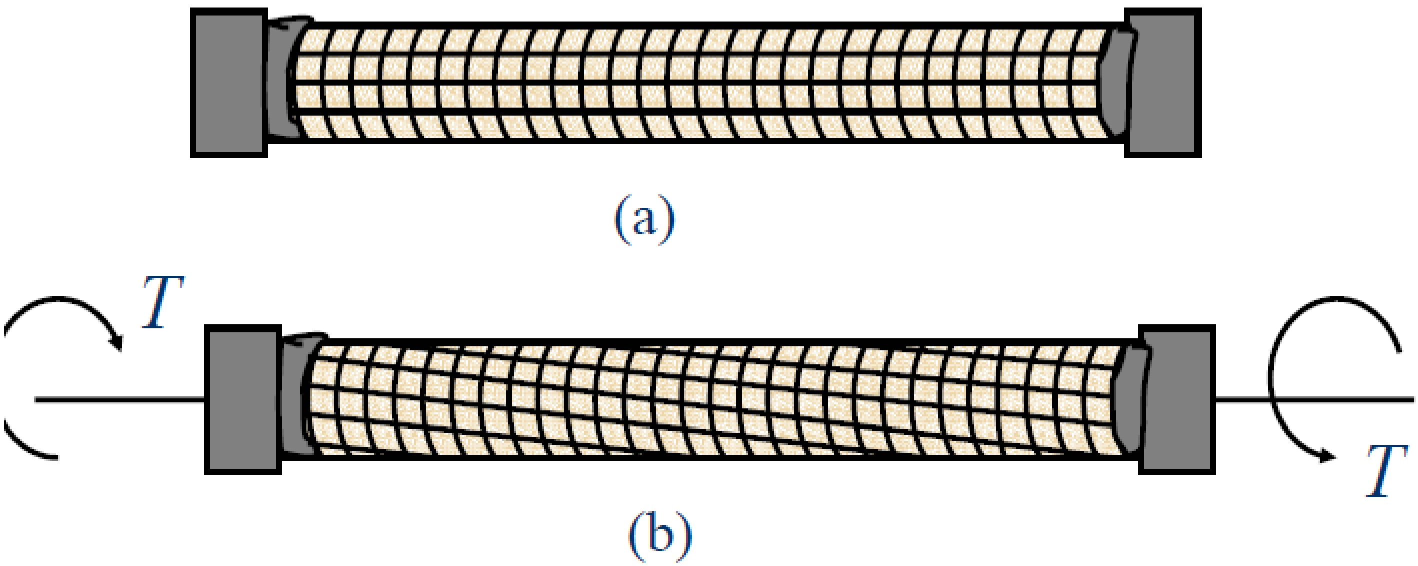
CASE I

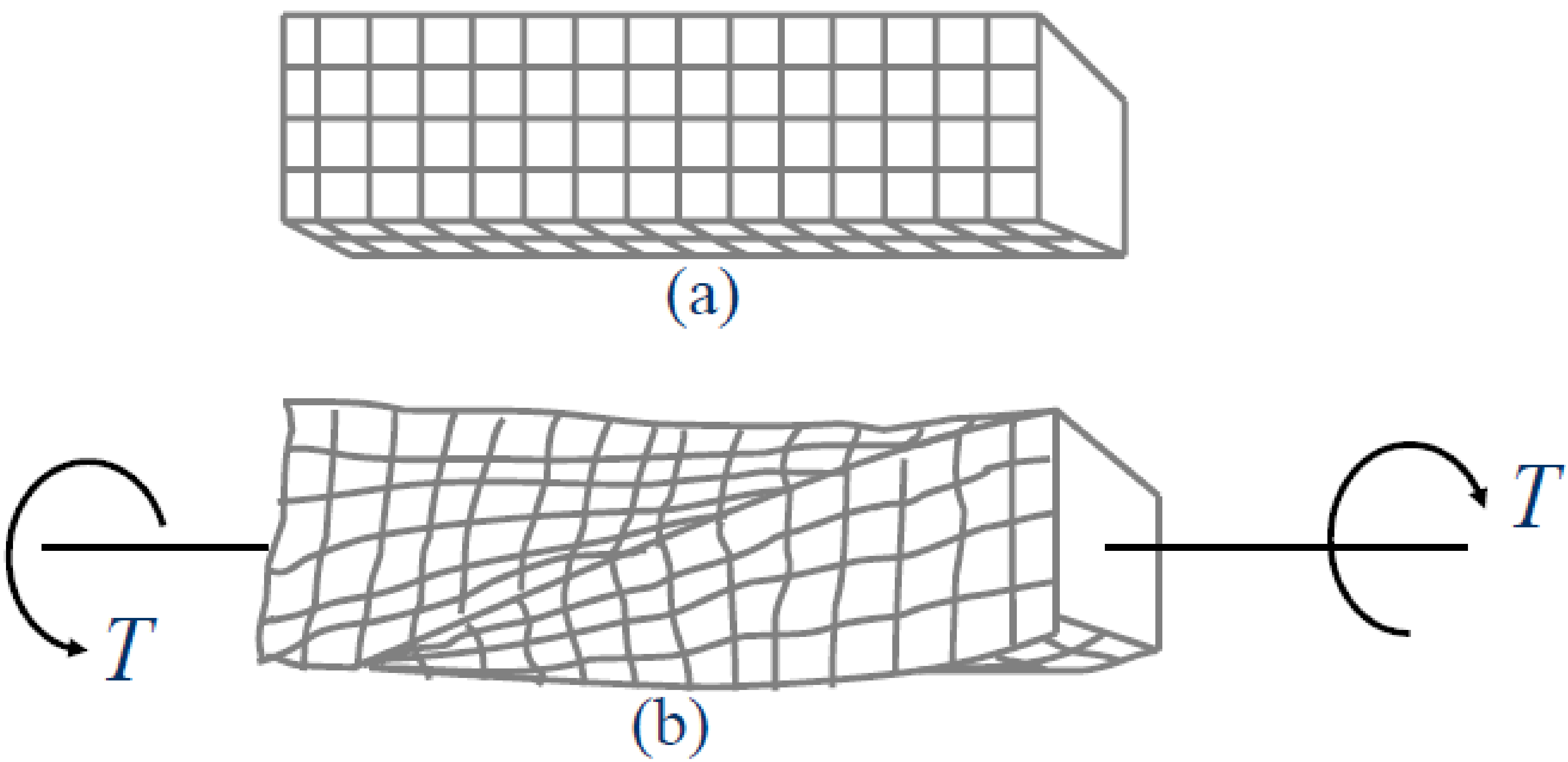
The first application area is the torsion of Non-Circular sections. The governing differential equations is

$$\frac{1}{G}\frac{\partial^2\phi}{\partial x^2}+\frac{1}{G}\frac{\partial^2\phi}{\partial y^2}+2\theta = 0$$

where G is the shear modulus of the material and θ is the angle of twist. The above Equation is obtained from equation (2) by noting that.

$$a_{11} = a_{22} = 1/G, \quad a_{00} = 0 \text{ and } f = 2\theta$$


(a)
T
T
(b)


(a)
T
(b)
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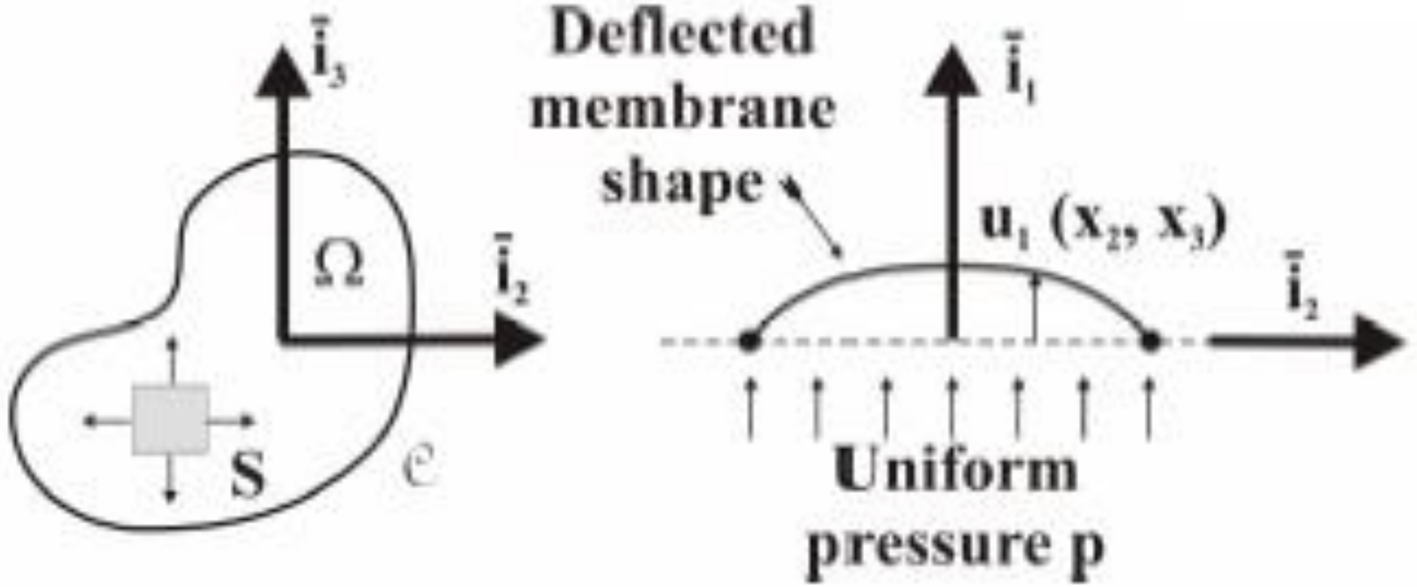
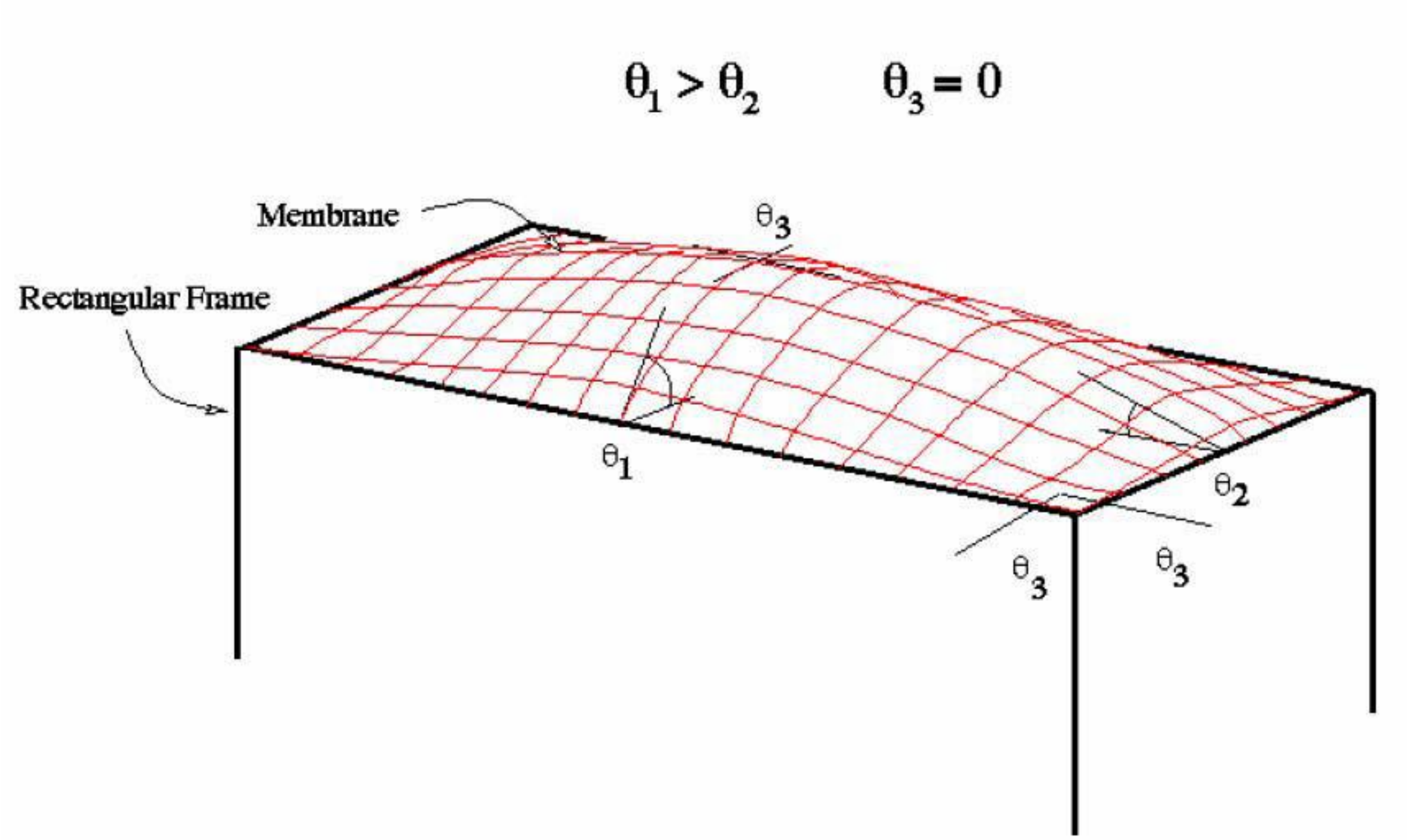
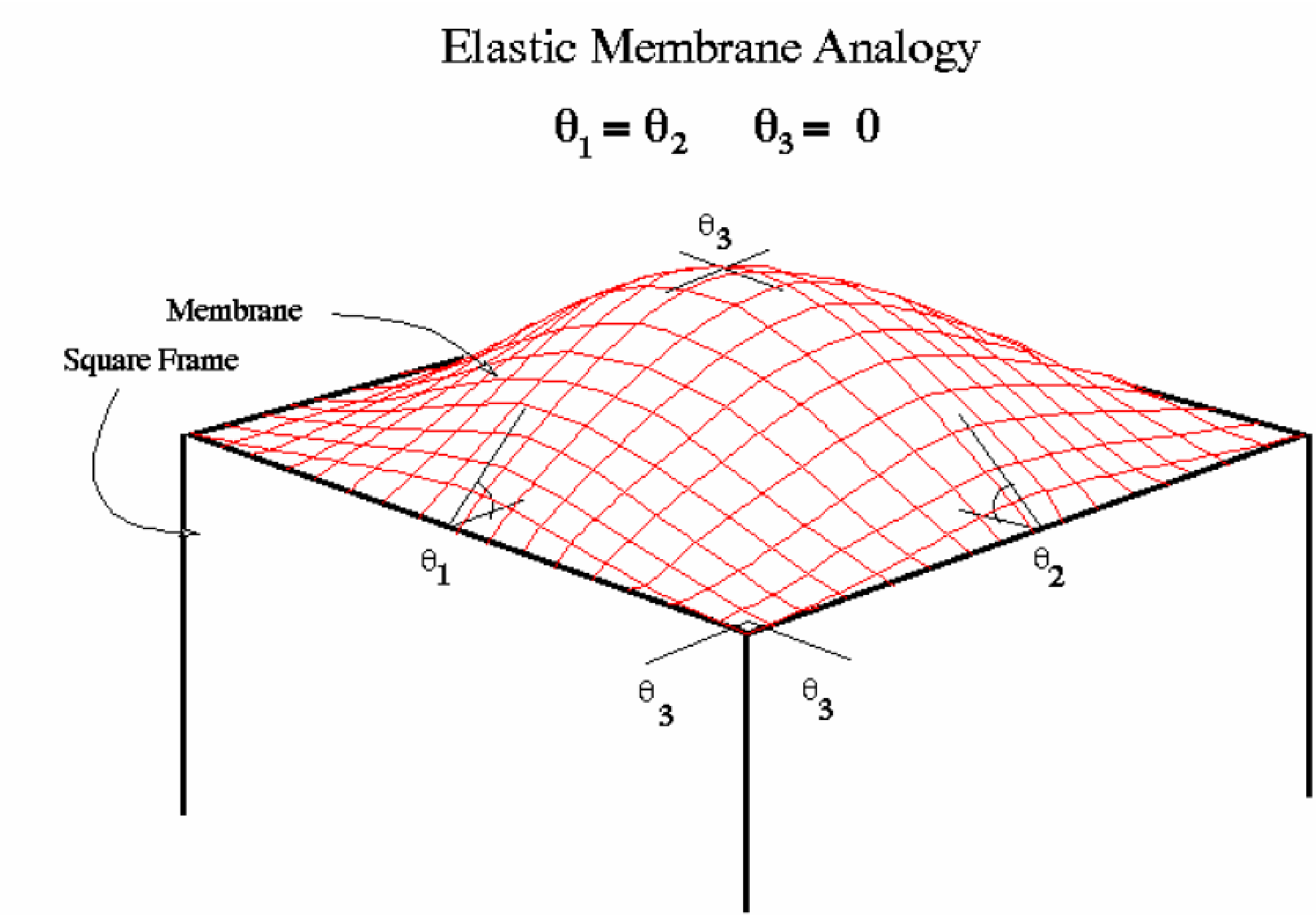


The thin membrane attached to the contour $\mathcal{C}$.

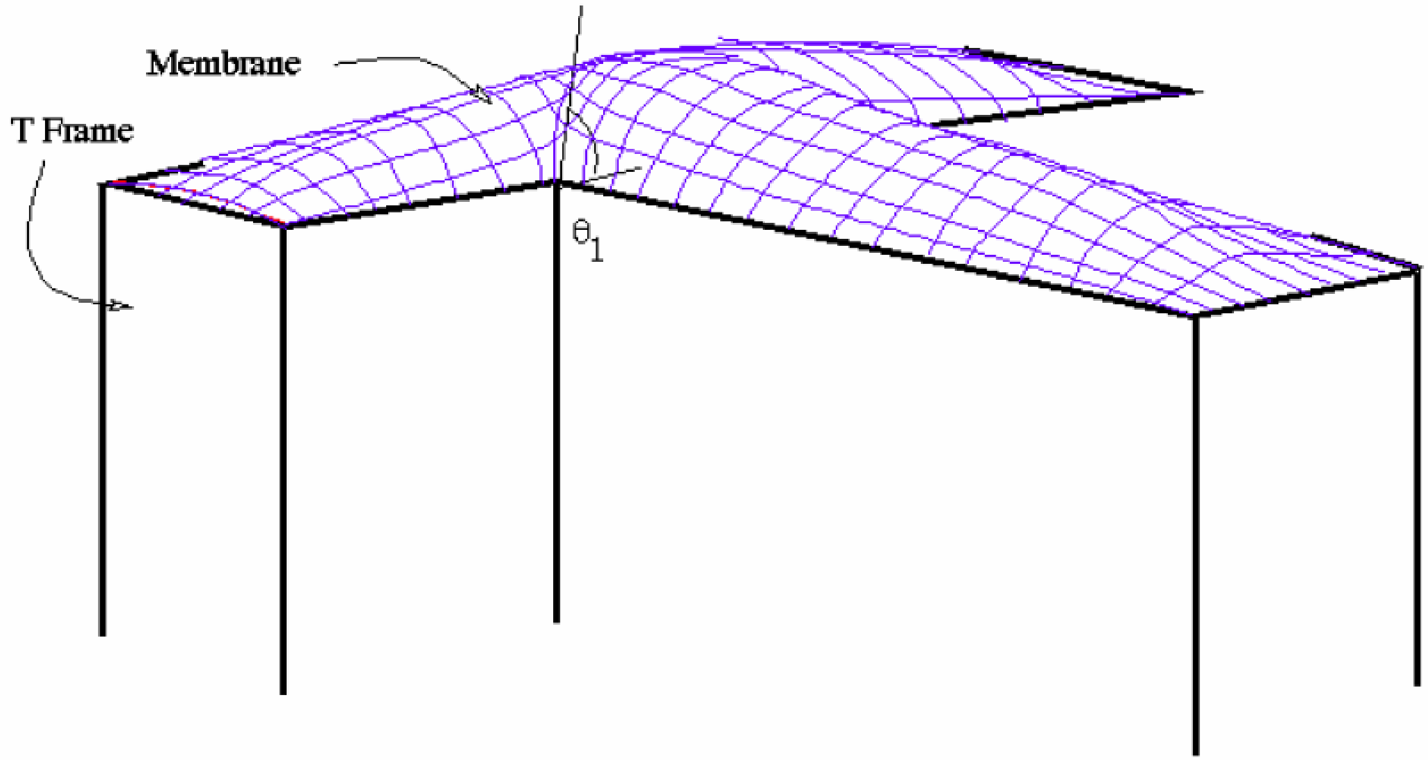


http://www.ae.msstate.edu/%7Emasoud/Teaching/SA2/A6.5_more2.html



http://www.ae.msstate.edu/%7Emasoud/Teaching/SA2/A6.5_more3.html

Elastic Membrane Analogy
$\theta_1 =$ Maximum



The variable ϕ is a stress function and the shear stresses within the shaft are related to the derivatives of ϕ with respect to x and y.

$$\tau_{zx} = \frac{\partial \phi}{\partial y} \quad \text{and} \quad \tau_{zy} = - \frac{\partial \phi}{\partial x}$$

On the free boundary $\phi = 0$. This is the case of a Poisson's Equation

CASE II

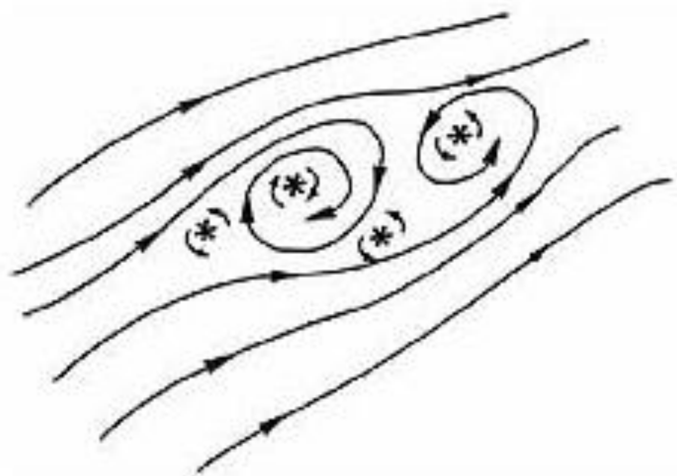
Several Fluid Mechanics Problems are embedded within equation (2). The streamline and potential formulations for an ideal irrotational fluid are governed by

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad \text{and}$$

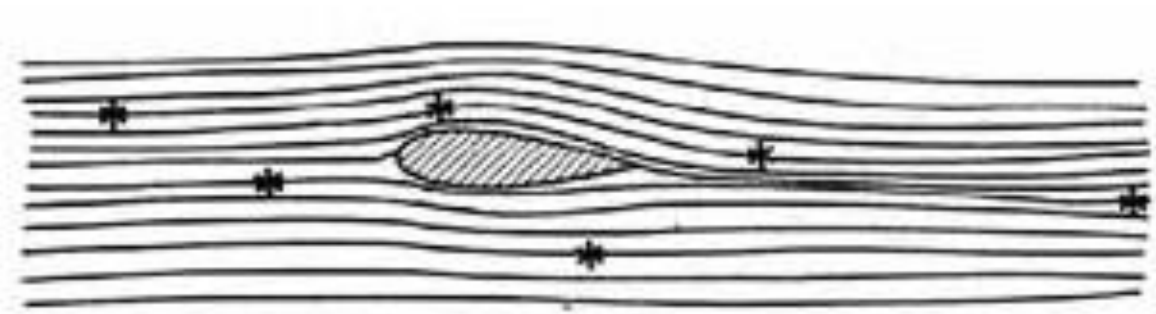
$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \qquad \text{respectively}$$

The streamlines ψ are perpendicular to the constant potential lines ϕ , and the velocity components are related to the derivatives of either ϕ or ψ with respect to x and y.

(a) Irrotational flow.



(b) Rotational flow.



(c) Inviscid, irrotational flow about an airfoil.

CASE III

The flow of water within the earth is governed by equations in (2). The seepage of water under a dam or retaining wall and with in a confined acqufier is given by

$$D_x \frac{\partial^2 \phi}{\partial x^2} + D_y \frac{\partial^2 \phi}{\partial y^2} = 0$$

Where D_x and D_y are the permeabilities of the earth material and ϕ represents the piezometric head.

The water level around a well during the pumping process is governed by

$$D_x \frac{\partial^2 \phi}{\partial x^2} + D_y \frac{\partial^2 \phi}{\partial y^2} + Q = 0$$

where Q is a point sink term

CASE IV

There are two heat transfer equations embedded with (2). The heat transfer from a 2-D fin to the surrounding fluid by convection is governed by

$$K_x \frac{\partial^2 T}{\partial x^2} + K_y \frac{\partial^2 T}{\partial y^2} - \frac{2h}{t} T - \frac{2h}{t} T\infty = 0$$

The coefficients K_x and K_y represent the thermal conductive coefficient in the x and y directions, respectively;

h is the convection coefficient; t is the thickness of the fin; T_∞ is the ambient temperature of the medium and T is the temperature of the fin.

If the fin is assumed to be thin and the heat loss from the edges is neglected. Then the equation becomes

$$K_x \frac{\partial^2 T}{\partial x^2} + K_y \frac{\partial^2 T}{\partial y^2} = 0$$

CASE V

A fluid vibrating within a closed volume is represented as

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{w^2}{c^2} P = 0$$

where P is the pressure excess above the ambient pressure, w is the wave frequency and c is the wave velocity in the medium.

CASE VI

When a_{00} is negative and ϕ equals zero, the differential equation is called a Helmholtz equation. A negative a_{00} yields an eigen value problem. Physical problems of Helmholtz equation is the wave motion for shallow bodies of water and Acoustical Vibrations in closed rooms

$$h\frac{\partial^2 w}{\partial x^2} + h\frac{\partial^2 w}{\partial y^2} + \frac{4\Pi^2}{g\,T^2}w = 0$$

.

Where,

h is water depth at the quiescent state

w is the wave height above the quiescent level

g is the gravitational constant and

T is the period of oscillations

CASE VII

In the area of electrical engineering, there are several interacting problems involving scalar and vector fields. In an isotropic dielectric medium with a permittivity ε (F/m), and a volume charge density ρ (C/m) the electric potential u (V) must satisfy the equation

$$\varepsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+\rho=0$$

The magnetic field problem is represented by

$$\mu \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

where

u is the scalar magnetic potential (A) and

μ is the permeability

Types of 2D Problems

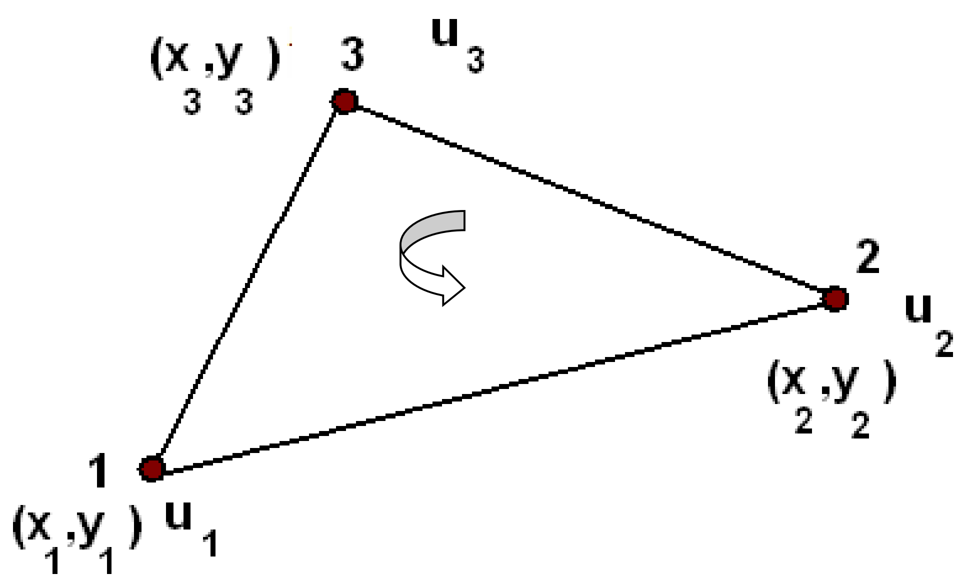
- **VECTOR VARIABLE PROBLEMS**

  e.g. Torsion of non-circular shafts, Heat transfer through fins

- **SCALAR VARIABLE PROBLEMS**

  e.g. Structural problems

Shape functions for three noded linear triangular element also called as Constant strain triangular(CST) element



1,2,3 Node numbers

u_1,u_2,u_3 Nodal value of field variable

(x_1,y_1), $(x_2,y_2)(x_3,y_3)$ nodal coordinates

Displacement model: $u(x,y) = a_1 + a_2 x + a_3 y$

$$u(x,y) = \langle 1 \quad x \quad y \rangle \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$u_1 = \mathrm{a}_1 + a_2 \mathrm{x}_1 + a_3 \mathrm{y}_1$$

$$u_2 = \mathrm{a}_1 + a_2 \mathrm{x}_2 + a_3 \mathrm{y}_2$$

$$u_3 = \mathrm{a}_1 + a_2 \mathrm{x}_3 + a_3 \mathrm{y}_3$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & \mathrm{x}_1 & \mathrm{y}_1 \\ 1 & \mathrm{x}_2 & \mathrm{y}_2 \\ 1 & \mathrm{x}_3 & \mathrm{y}_3 \end{bmatrix} \begin{Bmatrix} \mathrm{a}_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

i.e. $\{\mathrm{u}\}^e = [\mathrm{P}]\{\mathrm{a}\}^e$

The generalised coordinates are given in terms of nodal displacements as

$$\{a\}^e = [P]^{-1} \{u\}^e$$

provided $|P| \neq 0$ which is the area bounded by the three vertices.

Substituting for a_is in the displacement model

$$u(x,y) = \langle 1 \quad x \quad y \rangle \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$u(x,y) = \langle 1 \quad x \quad y \rangle \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \langle 1\text{x}3 \rangle [3\text{x}3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \langle N_1 N_2 N_3 \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u(x,y) = \sum_{j=1}^{3} N_i\,(x,y)\,u_j$$

where,

$$N_i\,(x,y) = \frac{1}{2A_e}\,(\alpha_i + \beta_i x + \gamma_i\, y)$$

$$\alpha_i = x_j\, y_k - x_k\, y_j$$

$$\beta_i = y_j - y_k$$

$$\gamma_i = -(\,x_j - x_k\,)$$

and *Here i, j, k permute in the natural order*

$$N_i\,(x,y) = \frac{1}{2A_e}\,(\alpha_i + \beta_i x + \gamma_i\, y)$$

$$N_1\,(x,y) = \frac{1}{2A_e}\,(\alpha_1 + \beta_1 x + \gamma_1\, y)$$

$$N_2\,(x,y) = \frac{1}{2A_e}\,(\alpha_2 + \beta_2 x + \gamma_2\, y)$$

$$N_3\,(x,y) = \frac{1}{2A_e}\,(\alpha_3 + \beta_3 x + \gamma_3\, y)$$

$$\alpha_i = x_j y_k - x_k y_j$$

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 2A_e$$

$$\alpha_1 = x_2 y_3 - x_3 y_2$$

$$\alpha_2 = x_3 y_1 - x_1 y_3$$

$$\alpha_3 = x_1 y_2 - x_2 y_1$$

$$\beta_i = y_j - y_k$$

$$\gamma_i = x_k - k_j$$

$$\beta_1 = y_2 - y_3$$

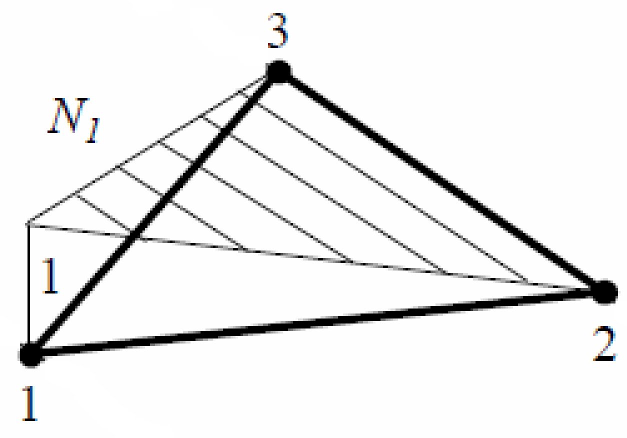
$$\gamma_1 = -(x_2 - x_3)$$

$$\beta_2 = y_3 - y_1$$

$$\gamma_2 = -(x_3 - x_1)$$

$$\beta_3 = y_1 - y_2$$

$$\gamma_3 = -(x_1 - x_2)$$

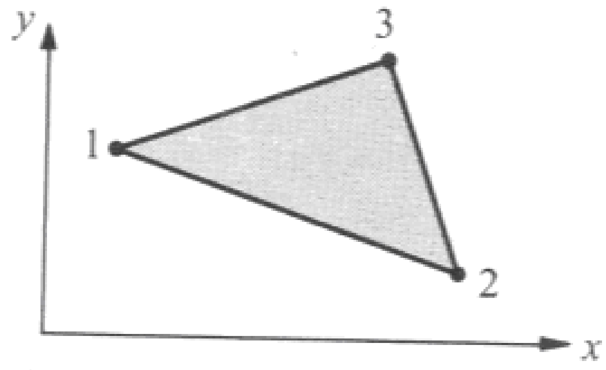


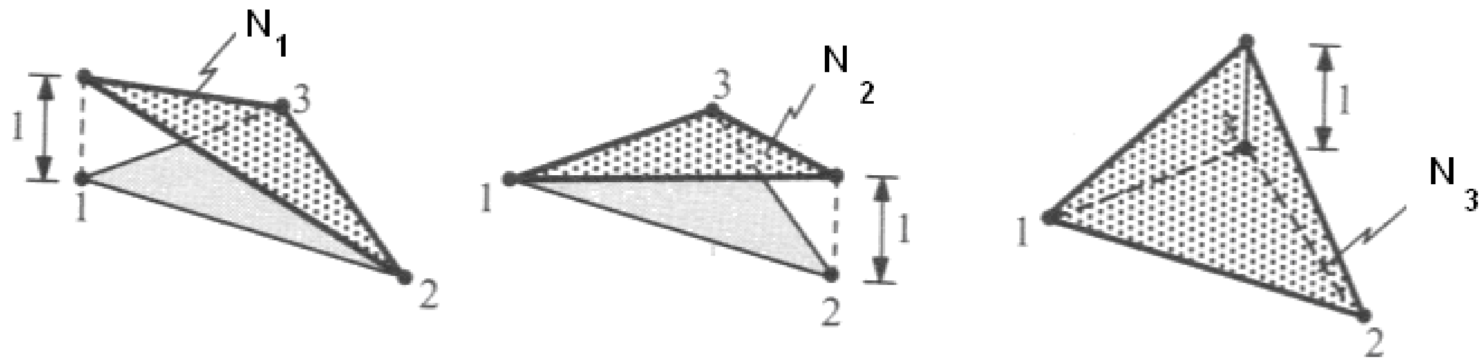
Shape Function N_1 for CST

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{\mathrm{yy}} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \varepsilon_{\mathrm{xy}} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

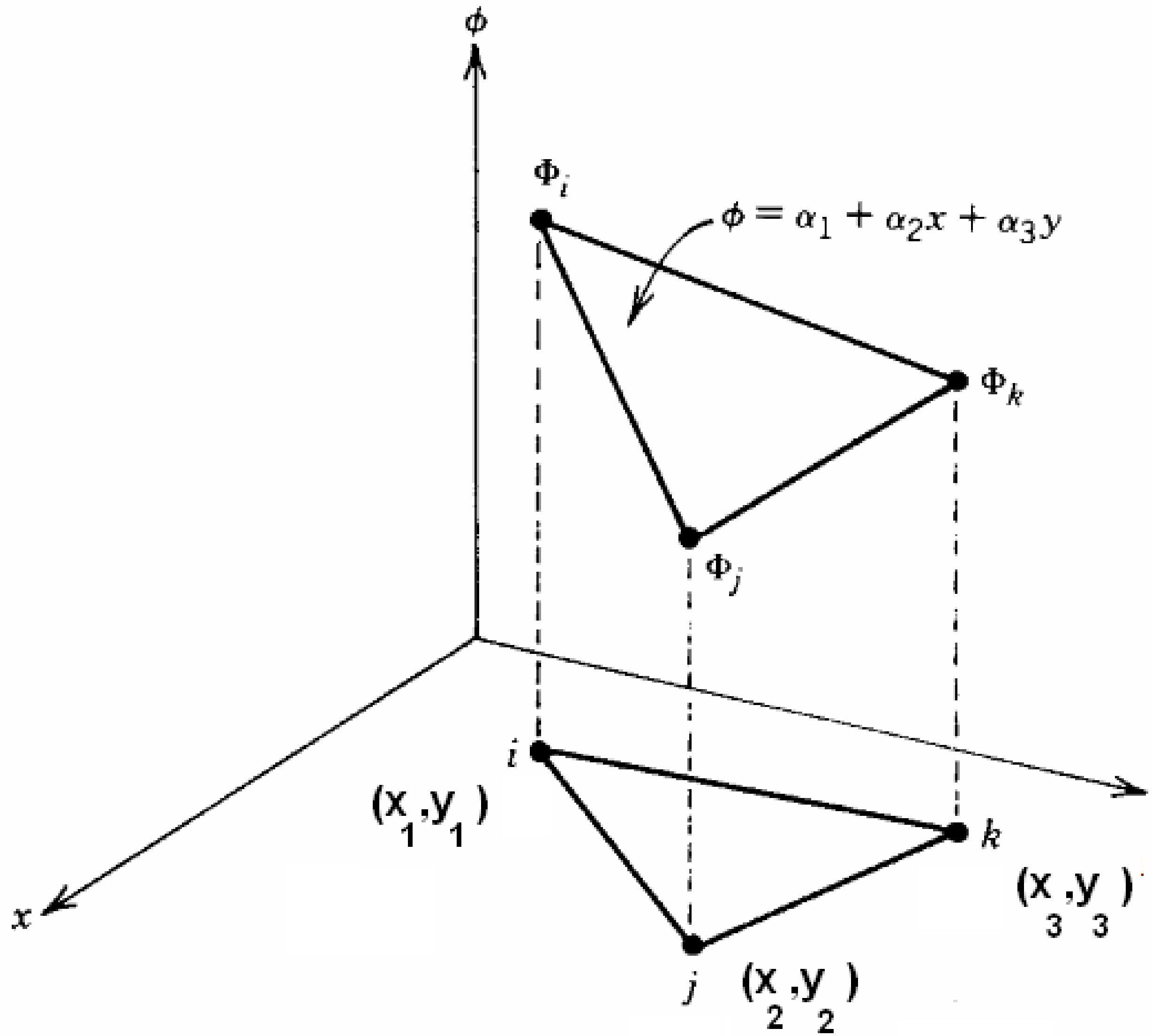
$$\varepsilon_{xx} = \frac{1}{2A}(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3)$$

$$\varepsilon_{\mathrm{yy}} = \frac{1}{2A}(\gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3)$$





Variation of Shape functions for CST element



ϕ
Φ_i
$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y$
Φ_k
Φ_j
i
(x_1,y_1)
y
k
(x_3,y_3)
x
j
(x_2,y_2)

Applications of the CST Element:

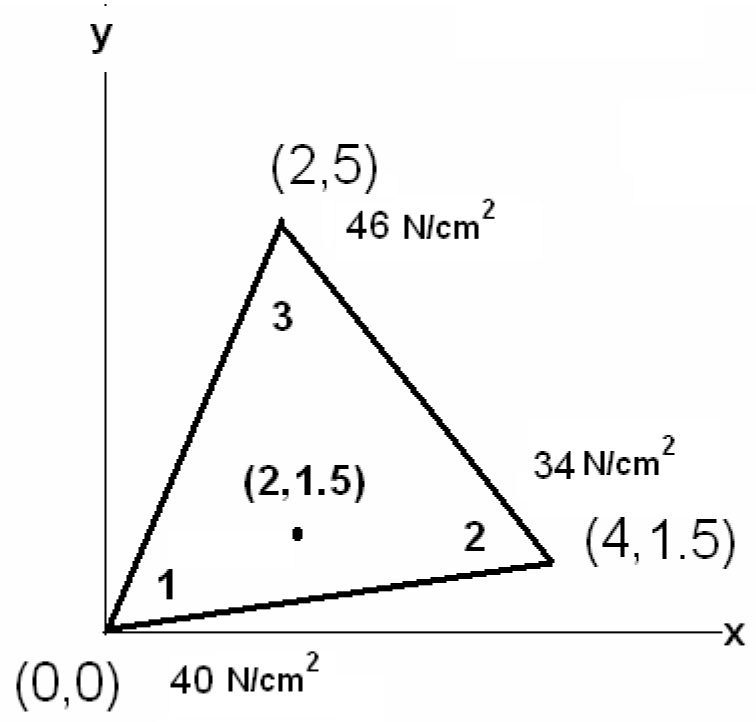
- ➢· Used in areas where the strain gradient is small.
- ➢· Used in mesh transition areas (fine mesh to coarse mesh).
- ➢· Use of CST in stress concentration or other crucial areas in the structure, such as edges of holes and corners is to be avoided
- ➢· Recommended for quick and preliminary FE analysis of 2-D problems

<u>Problem1</u>:- Given the nodal values of pressure in a triangular element as $P_1 = 40$ N/cm^2, $P_2 = 34$ N/cm^2 & $P_3 = 46$ N/cm^2 evaluate the element shape functions and calculate the value of the pressure at a point whose co-ordinates are given by (2, 1.5). The co-ordinates of nodes 1, 2 & 3 are respectively (0,0), (4, 1.5), (2,5).

$$\alpha_1 = x_2y_3 - x_3y_2 = 19$$
$$\alpha_2 = x_3y_1 - x_1y_3 = 0$$
$$\alpha_3 = x_1y_2 - x_2y_1 = 0$$

$$\beta_1 = y_2 - y_3 = -4.5$$
$$\beta_2 = y_3 - y_1 = 5$$
$$\beta_3 = y_1 - y_2 = -0.5$$

$$\gamma_1 = -(x_2 - x_3) = -2$$
$$\gamma_2 = -(x_3 - x_1) = -2$$
$$\gamma_3 = -(x_1 - x_2) = 4$$



$$2A = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 4 & 0.5 \\ 1 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 19 \text{ cm}^2$$

$$N_1 = \frac{1}{2A}(\alpha_1 + \beta_1 x + \gamma_1 y) = \frac{1}{19}(19 - 4.5x - 2y)$$

$$N_2 = \frac{1}{2A}(\alpha_2 + \beta_2 x + \gamma_2 y) = \frac{1}{19}(5x - 2y)$$

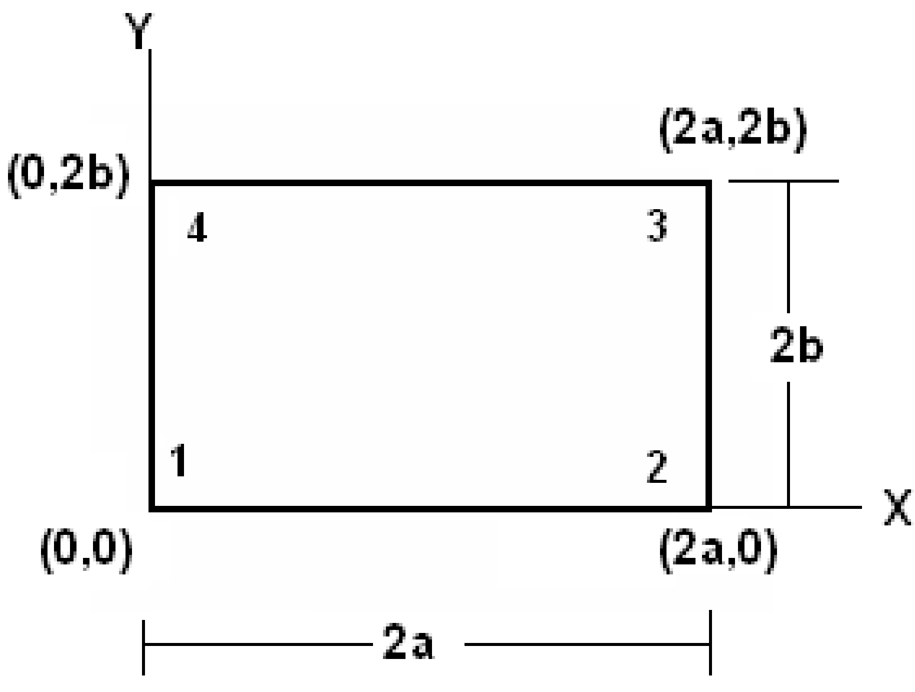
$$N_3 = \frac{1}{2A}(\alpha_3 + \beta_3 x + \gamma_3 y) = \frac{1}{19}(-0.5x + 4y)$$

$$\text{Now } P(x, y) = N_1P_1 + N_2P_2 + N_3P_3$$
$$= 1/19\ [(19 - 4.5x - 2y)\ 40 + (5x - 2y)\ 34 - (0.5x - 4y)\ 46]$$

$$\therefore P\ (2, 15) = 14.74 + 12.53 + 12.11$$
$$= 39.37\ \text{N/cm}^2$$

BI – LINEAR RECTANGULAR ELEMENT

Cartesian co-ordinates (generalized co-ordinates)



Let the assumed displacement model be given by

$$u(x,y) = c_0 + c_1 x + c_2 y + c_3 xy \quad \text{----} \ (1)$$

$$= \langle 1 \;\; x \;\; y \;\; xy \rangle \begin{Bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{Bmatrix}$$

Let u_1, u_2, u_3 & u_4 represent the nodal values of the field variable at nodes 1, 2, 3 & 4. Substituting the respective x, y co-ordinates of the nodes we get

$$u_1 = c_0$$
$$u_2 = c_0 + 2a\, c_1$$
$$u_3 = c_0 + 2a\, c_1 + 2b\, c_2 + 4ab\, c_3$$
$$u_4 = c_0 + 2b\, c_2$$

or

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2a & 0 & 0 \\ 1 & 2a & 2b & 4ab \\ 1 & 0 & 2b & 0 \end{pmatrix} \begin{Bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \quad \text{---- (2)}$$

Here c_i represents the generalised co-ordinates which can be obtained by

$$\begin{Bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2a & 0 & 0 \\ 1 & 2a & 2b & 4ab \\ 1 & 0 & 2b & 0 \end{pmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \quad \text{-----(3)}$$

Substituting (3) in (1) we get

$$u(x, y) = \underbrace{\underset{1 \times 4}{\langle 1 \quad x \quad y \quad xy \rangle} \underset{4 \times 4}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2a & 0 & 0 \\ 1 & 2a & 2b & 4ab \\ 1 & 0 & 2b & 0 \end{pmatrix}^{-1}}}_{(1 \times 4)} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$= \langle N_1 \quad N_2 \quad N_3 \quad N_4 \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

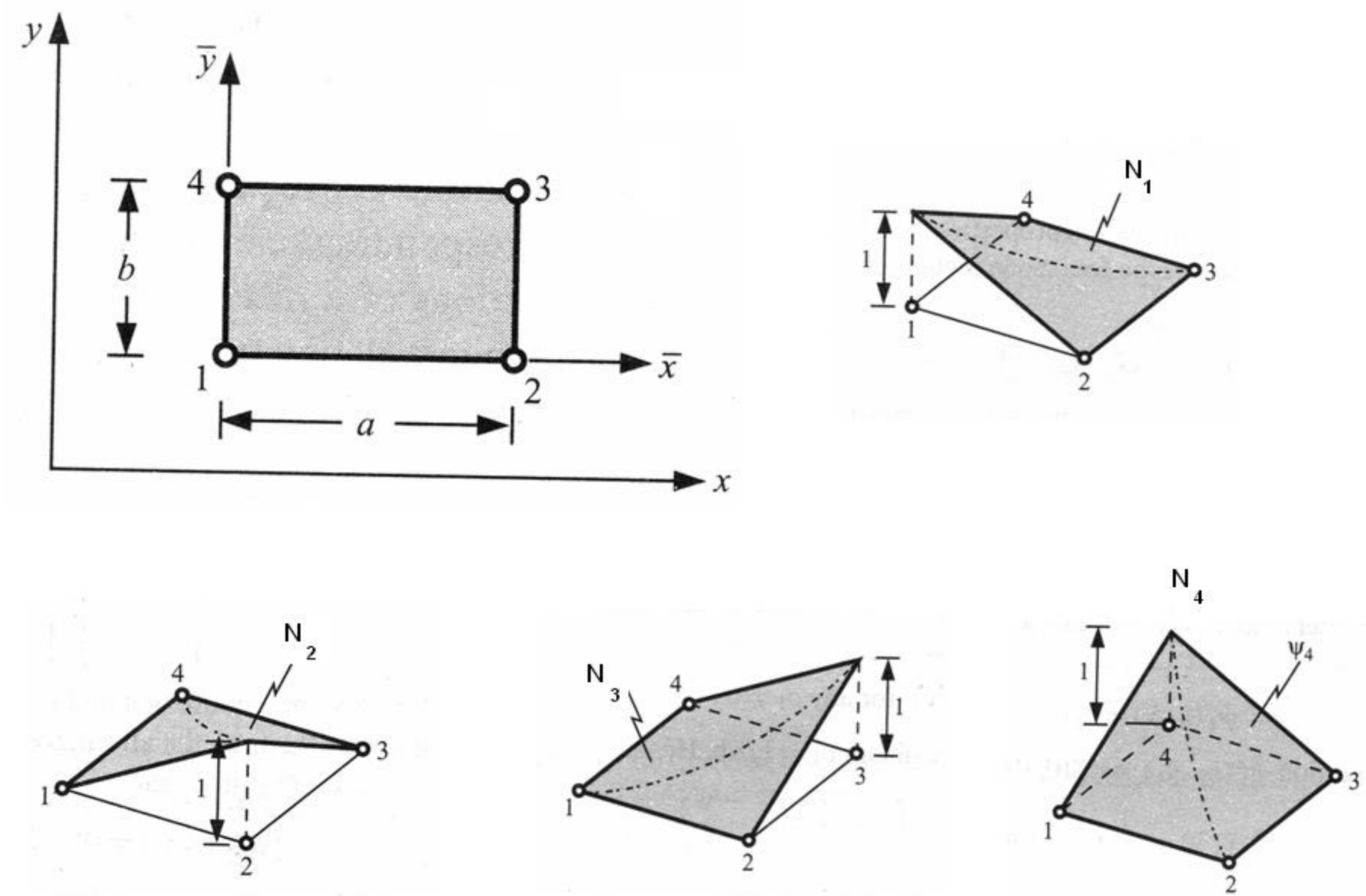
where

$$N_1 = \left(1 - \frac{x}{2a}\right)\left(1 - \frac{y}{2b}\right)$$

$$N_2 = \frac{x}{2a}\left(1 - \frac{y}{2b}\right)$$

$$N_3 = \frac{x}{2a}\,\frac{y}{2b} = \frac{xy}{4ab}$$

$$N_4 = \left(1 - \frac{x}{2a}\right)\frac{y}{2b}$$



y
ȳ
4
3
b
1
2
x̄
a
x
N1
N2
N3
N4
ψ4

LAGRANGIAN INTERPOLATION POLYNOMIALS: (CARTESIAN CO-ORDINATES)

$$N_1(x,y) = N_1(x)N_1(y) = \frac{(x-x_2)}{(x_1-x_2)}\frac{(y-y_4)}{(y_1-y_4)} = \left(\frac{x-2a}{0-2a}\right)\left(\frac{y-2b}{0-2b}\right)$$

$$= \left(1-\frac{x}{2a}\right)\left(1-\frac{y}{2b}\right)$$

$$N_2(x,y) = N_2(x)N_2(y) = \frac{(x-x_1)}{(x_2-x_1)}\frac{(y-y_3)}{(y_2-y_3)} = \left(\frac{x-0}{2a-0}\right)\left(\frac{y-2b}{0-2b}\right)$$

$$= \left(\frac{x}{2a}\right)\left(1-\frac{y}{2b}\right)$$

$$N_3(x,y) = N_3(x)N_3(y) = \frac{(x-x_4)}{(x_3-x_4)}\frac{(y-y_2)}{(y_3-y_2)} = \left(\frac{x-0}{2a-0}\right)\left(\frac{y-0}{2b-0}\right)$$

$$= \left(\frac{xy}{4ab}\right)$$

$$N_4(x,y) = N_4(x)N_4(y) = \frac{(x-x_3)}{(x_4-x_3)}\frac{(y-y_1)}{(y_4-y_1)} = \left(\frac{x-2a}{0-2a}\right)\left(\frac{y-0}{2b-0}\right)$$

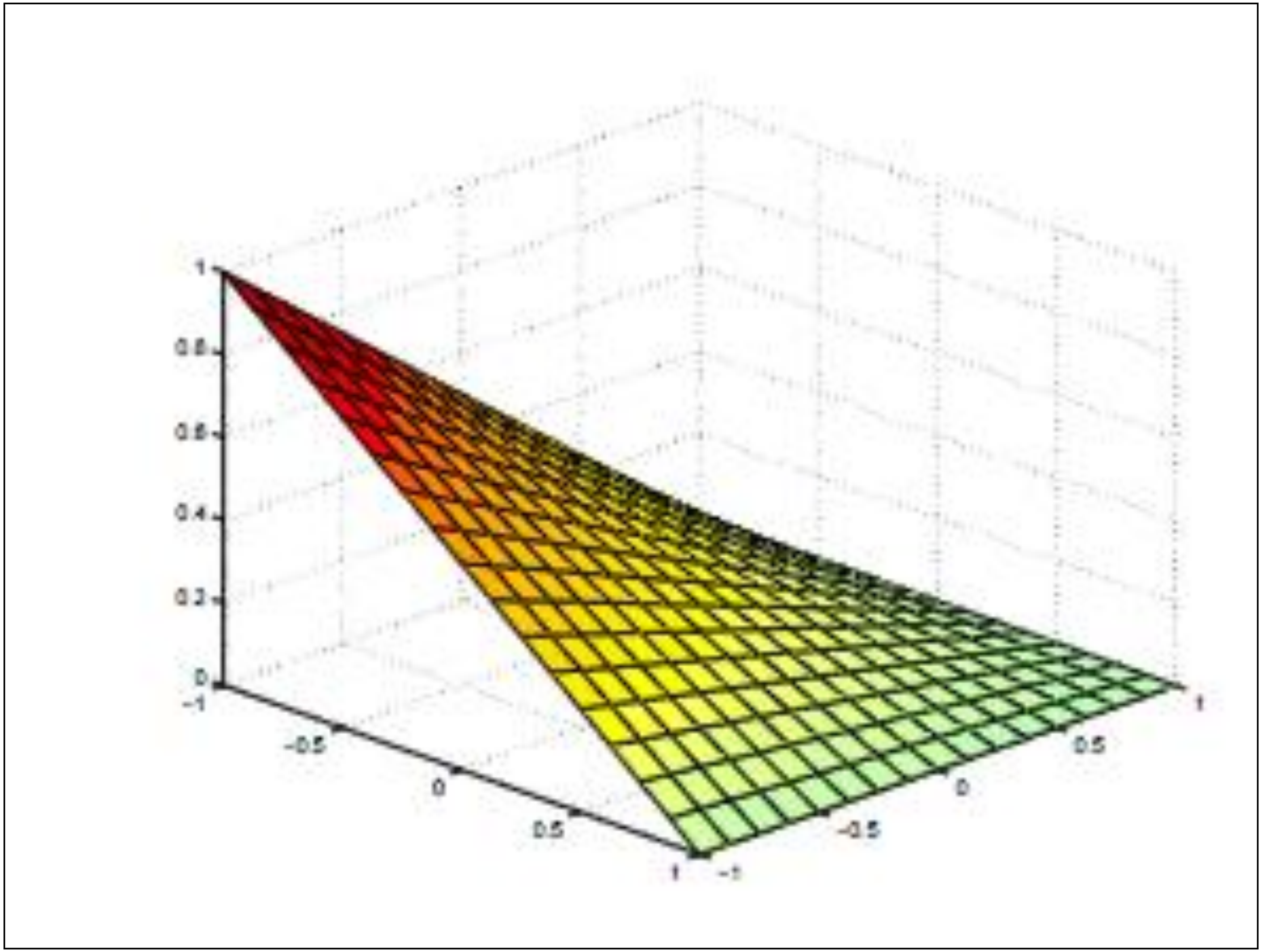
$$= \left(\frac{y}{2b}\right)\left(1-\frac{x}{2a}\right)$$

It should be noted here that $\sum_{i=1}^{n} N_i = 1$ at any point in the element.

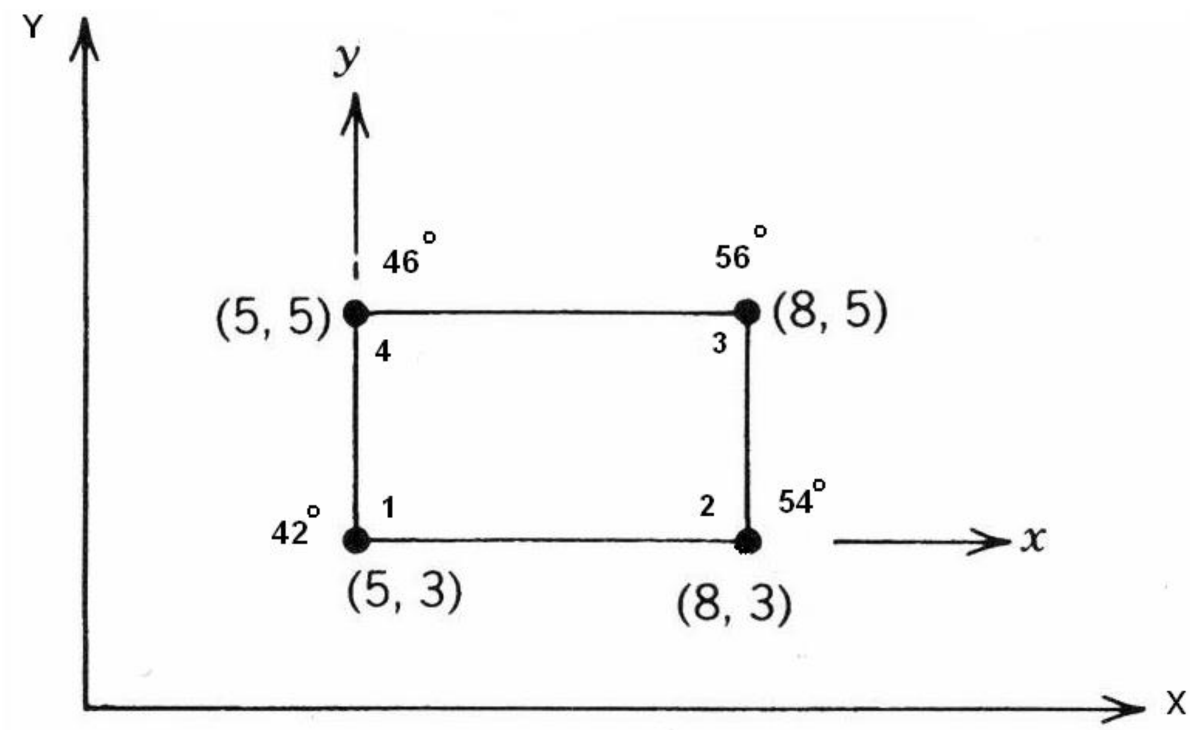
The variation of field variable over the element of bilinear element is given by

$$u(x, y) = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4$$

$$= \sum_{i=1}^{4} N_iu_i$$


1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
-1
-0.5
0
0.5
1

Determine three points on the 50°C contour line for the rectangular element shown the Fig. The nodal values are T_1 = 42°C, T_2 = 54°C, T_3 = 56°C, and T_4 = 46°C.



Nodal Coordinates

The length of the sides are

$2b = X_2 - X_1 = 8 - 5 = 3$

$2a = Y_4 - Y_1 = 5 - 3 = 2$

Substituting these values in the shape functions

$$N_1 = \left(1 - \frac{x}{3}\right)\left(1 - \frac{y}{2}\right) \qquad N_3 = \frac{xy}{6}$$

$$N_2 = \frac{x}{3}\left(1 - \frac{y}{2}\right) \qquad N_4 = \frac{y}{2}\left(1 - \frac{x}{3}\right)$$

Inspection reveals that the 50°C contour line intersects the sides *3-4* and 1-2; therefore, we need to assume values of y and calculate values of x. Along side *1-2*, y=0 and

$$T(x,y) = \left(1 - \frac{x}{3}\right)T_1 + \frac{x}{3}T_2 = 50$$

Substituting for T_1 and T_2 and solving gives x=2.0. Along side 4-2, y= 2a =2 and

$$T(x,y) = \frac{x}{3}T_4 + \left(1 - \frac{y}{3}\right)T_3 = 50$$

Substituting for T_4 and T_3 and solving gives x = 1.2

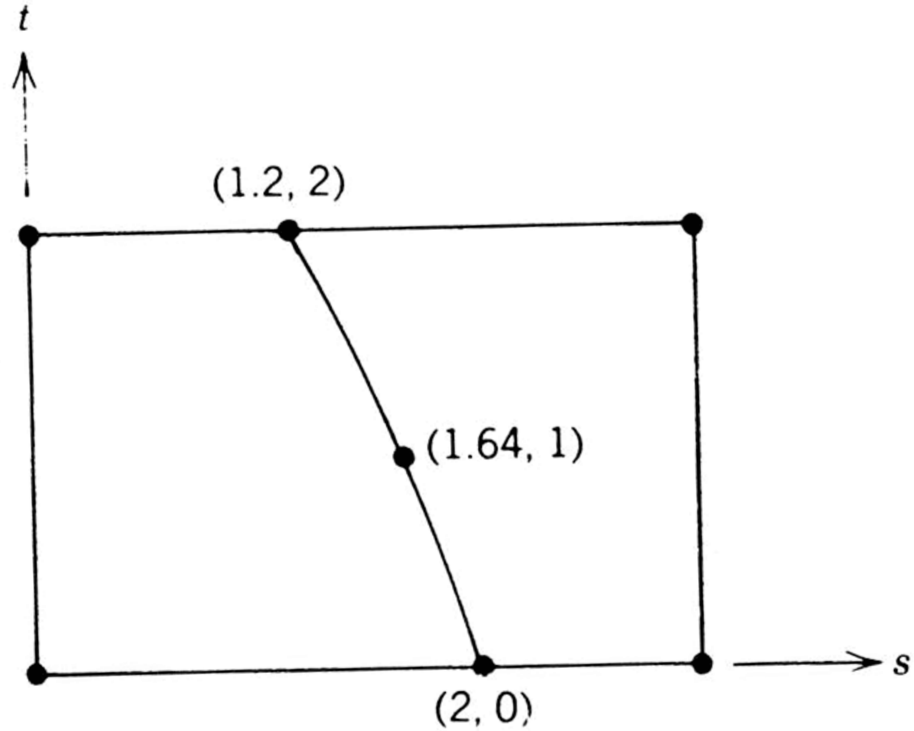
To obtain the third point, assume that y=a=1, then

$$T(x,y) = \frac{1}{2}\left(1 - \frac{x}{3}\right)T_1 + \frac{x}{6}T_2 + \frac{x}{6}T_3 + \frac{1}{2}\left(1 - \frac{x}{3}\right)T_4 = 50$$

Substituting the nodal values gives

$$\frac{x}{6}\left(-42 + 54 + 56 - 46\right) + \frac{1}{2}(42 + 46) = 50$$

Solving yields x= 1.64



The *xy* coordinates of the three points are (1.2,2), (1.64,1) and (2,0). The *XY* coordinates of these points are (6.2,5), (6.64,4) and (7,3). A straight line from (6.2,5) to (7,3) passes through the point (6.60,4); therefore, the contour line is not straight.

Torsion of Non-circular shaft:

The governing equation for the torsion problem is given by

$$\frac{1}{G}\frac{\partial^2\phi}{\partial x^2} + \frac{1}{G}\frac{\partial^2\phi}{\partial y^2} + 2\theta = 0$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2G\theta$$

$$\tau_{zx} = \frac{\partial\phi}{\partial y} \qquad \tau_{zy} = -\frac{\partial\phi}{\partial x}$$

On the free boundary $\phi = 0$.

To derive the weak form multiply the equation with a weighting function w(x,y)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2G\theta = 0$$

$$\iint \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2G\theta\right) w(x,y)\,dxdy = 0$$

$$\iint \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) w(x,y)\,dxdy + \iint 2G\theta\, w(x,y)\,dxdy = 0$$

$$\iint\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)w(x,y)\,dxdy + \iint 2G\theta\, w(x,y)\,dxdy = 0$$

$$\iint\frac{\partial^2\phi}{\partial x^2}w(x,y)\,dxdy + \iint\frac{\partial^2\phi}{\partial y^2}w(x,y)\,dxdy + \iint 2G\theta\, w(x,y)\,dxdy = 0$$

$$\oint w(x,y)\frac{\partial\phi}{\partial x}n_x - \iint\frac{\partial\phi}{\partial x}\frac{\partial w}{\partial x}\,dxdy$$

$$+\oint w(x,y)\frac{\partial\phi}{\partial y}n_y - \iint\frac{\partial\phi}{\partial y}\frac{\partial w}{\partial y}\,dxdy + \iint 2G\theta\, w(x,y)\,dxdy = 0$$

where n_x and n_y are the components (direction cosines) of the unit normal vector

As Φ is specified along the boundaries w(x,y) = 0 and the boundary terms vanish. The weak form becomes

$$\iint \frac{\partial \phi}{\partial x}\frac{\partial w}{\partial x}\,dxdy + \iint \frac{\partial \phi}{\partial y}\frac{\partial w}{\partial y}\,dxdy = \iint 2G\theta\, w(x,y)dxdy$$

Assuming a CST element and substituting Φ as $N_1\Phi_1 + N_2\Phi_2 + N_3\Phi_3$ and w(x,y) as $\boldsymbol{N_1}$, $\boldsymbol{N_2}$, $\boldsymbol{N_3}$ we get a system of 3 equations in 3 unknowns which can be written as

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}$$

Where

$$K_{ij} = \iint \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} dxdy + \iint \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} dxdy$$

$$f_j = \iint 2G\theta \, N_j dxdy = 0$$

$$K_{ij} = \iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\,dxdy + \iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\,dxdy$$

$$K_{11} = \iint \frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x}\,dxdy + \iint \frac{\partial N_1}{\partial y}\frac{\partial N_1}{\partial y}\,dxdy$$

$$\boxed{N_i\,(x,y) = \frac{1}{2A_e}\,(\alpha_i + \beta_i x + \gamma_i\, y)}$$

$$K_{11} = \frac{1}{4A^2} \iint \beta_1 \beta_1 dxdy + \frac{1}{4A^2} \iint \gamma_1 \gamma_1 dxdy$$

$$= \frac{1}{4A^2} \beta_1 \beta_1 \iint dxdy + \frac{1}{4A^2} \gamma_1 \gamma_1 \iint dxdy$$

$$= \frac{1}{4A} \left(\beta_1 \beta_1 + \gamma_1 \gamma_1 \right)$$

$$K_{12} = \frac{1}{4A^2} \iint \beta_1 \beta_2 dxdy + \frac{1}{4A^2} \iint \gamma_1 \gamma_2 dxdy$$

$$= \frac{1}{4A} \left(\beta_1 \beta_2 + \gamma_1 \gamma_2 \right)$$

$$K_{13} = \frac{1}{4A^2}\iint \beta_1\beta_3 dxdy + \frac{1}{4A^2}\iint \gamma_1\gamma_3 dxdy$$

$$= \frac{1}{4A}\left(\beta_1\beta_3 + \gamma_1\gamma_3\right)$$

$$K_{22} = \frac{1}{4A^2}\iint \beta_2\beta_2 dxdy + \frac{1}{4A^2}\iint \gamma_2\gamma_2 dxdy$$

$$= \frac{1}{4A}\left(\beta_2\beta_2 + \gamma_2\gamma_2\right)$$

$$K_{23} = \frac{1}{4A^2}\iint \beta_2\beta_3 dxdy + \frac{1}{4A^2}\iint \gamma_2\gamma_3 dxdy$$

$$= \frac{1}{4A}\left(\beta_2\beta_3 + \gamma_2\gamma_3\right)$$

$$K_{33} = \frac{1}{4A^2}\iint \beta_3\beta_3 dxdy + \frac{1}{4A^2}\iint \gamma_3\gamma_3 dxdy$$

$$= \frac{1}{4A}\left(\beta_3\beta_3 + \gamma_3\gamma_3\right)$$

$$[K] = \frac{1}{4A}\begin{bmatrix} \beta_1^2 + \gamma_1^2 & \beta_1\beta_2 + \gamma_1\gamma_2 & \beta_1\ \beta_3 + \gamma_1\gamma_3 \\ & \beta_1^2 + \gamma_2^2 & \beta_2\beta_3 + \gamma_2\gamma_3 \\ & & \beta_3^2 + \gamma_3^2 \end{bmatrix}$$

$$f_j = \iint 2G\theta\, N_j\, dxdy = 0$$

$$f_1 = \iint 2G\theta\, N_1\, dxdy = 0$$

$$= 2G\theta \frac{A}{3}$$

$$f_2 = \iint 2G\theta\, N_2\, dxdy = 0$$

$$= 2G\theta \frac{A}{3}$$

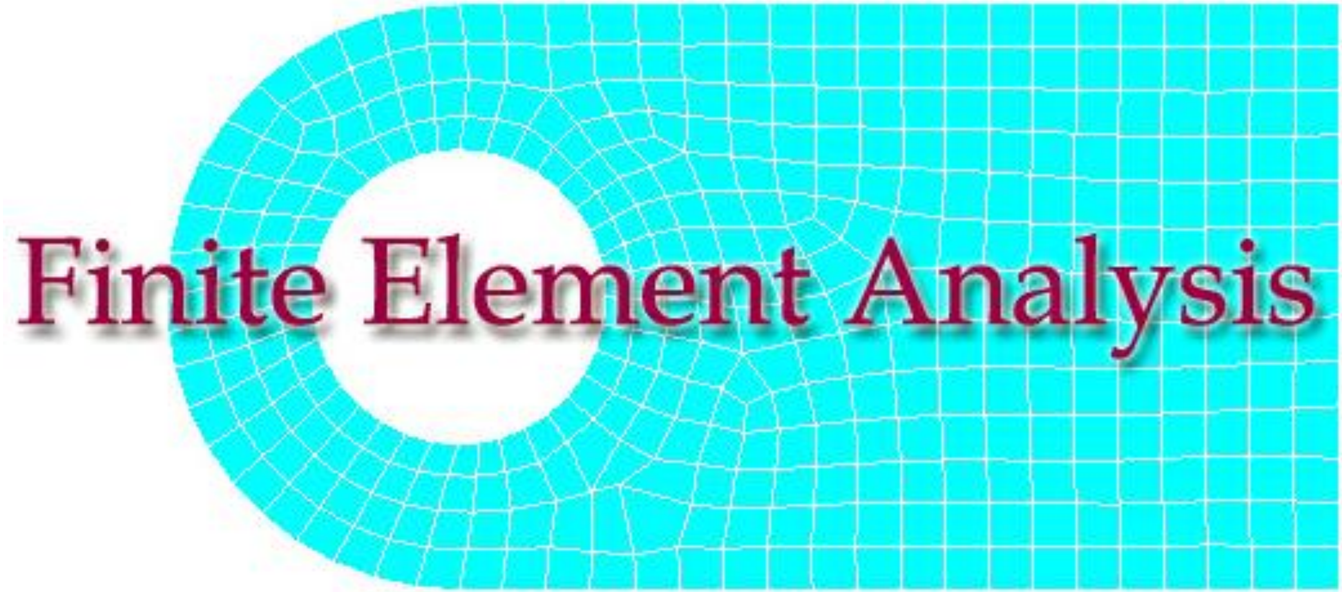
$$f_3 = \iint 2G\theta \, N_3 \, dxdy = 0$$

$$= 2G\theta \frac{A}{3}$$

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = 2G\theta \frac{A}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

2D linear elements
Linear triangular elements
Bi linear rectangular elements

Shape functions
Weak form for torsion problem
Simple problems



TWO DIMENSIONAL ELEMENTS

LECTURE 8

Types of 2D Problems

- **VECTOR VARIABLE PROBLEMS**

  e.g. Structural problems

- **SCALAR VARIABLE PROBLEMS**

  e.g. Torsion of non-circular shafts,

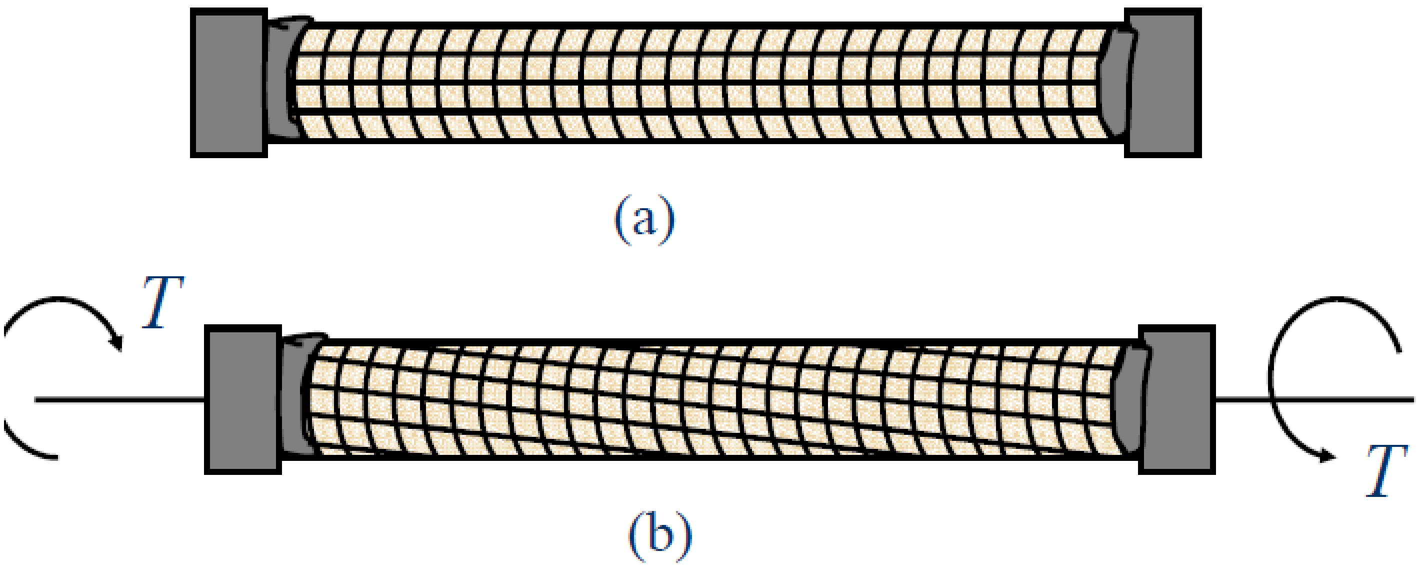
  Heat transfer through fins

The first application area is the **torsion of Non-Circular sections**. The governing differential equation is

$$\frac{1}{G}\frac{\partial^2\phi}{\partial x^2}+\frac{1}{G}\frac{\partial^2\phi}{\partial y^2}+2\theta = 0$$

where G - shear modulus of the material

θ - is the angle of twist.


(a)
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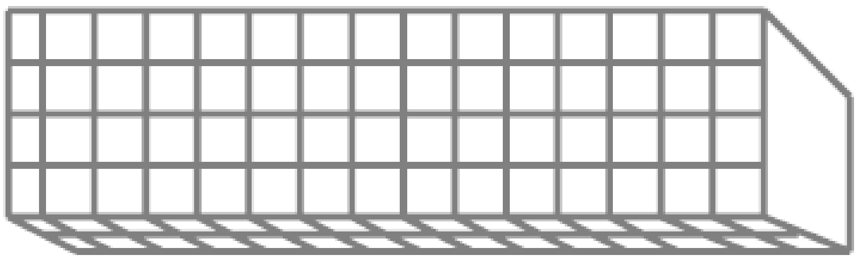
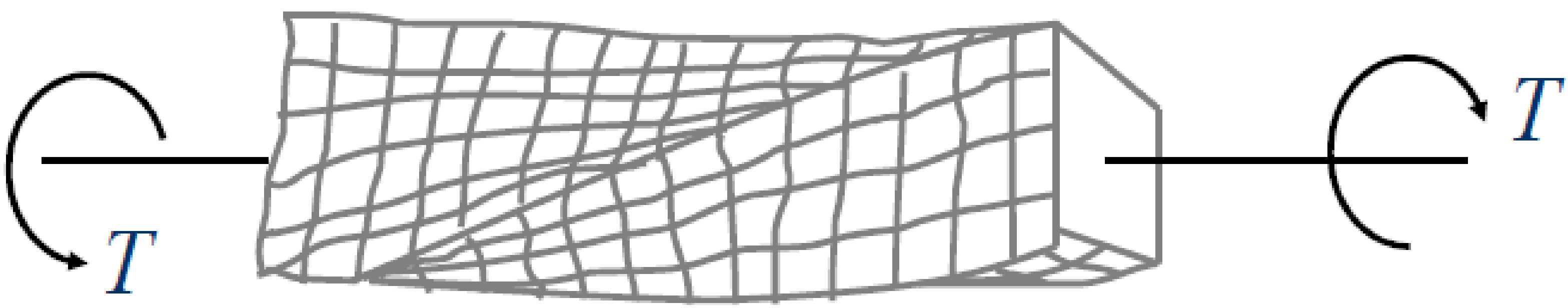


(a)



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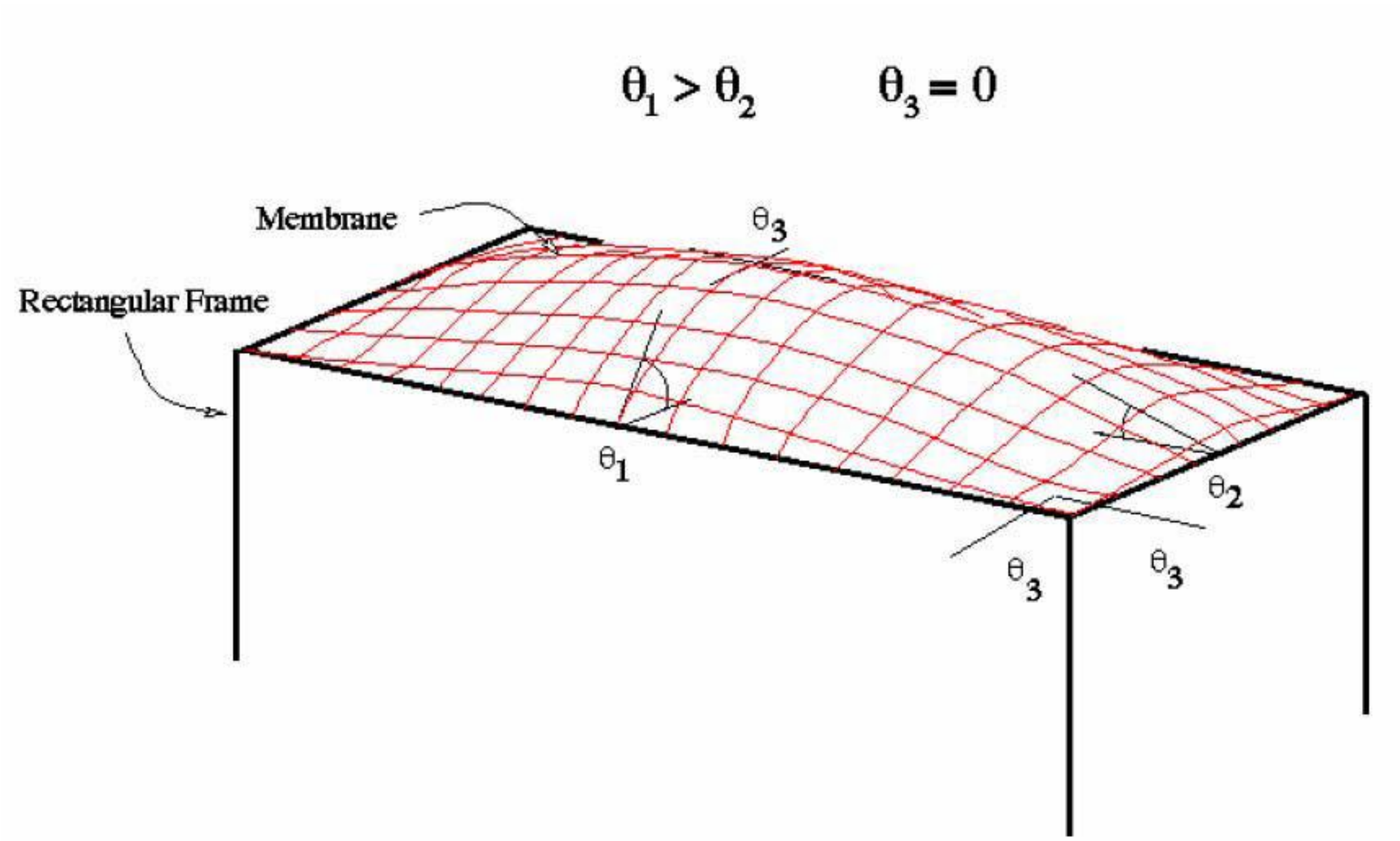
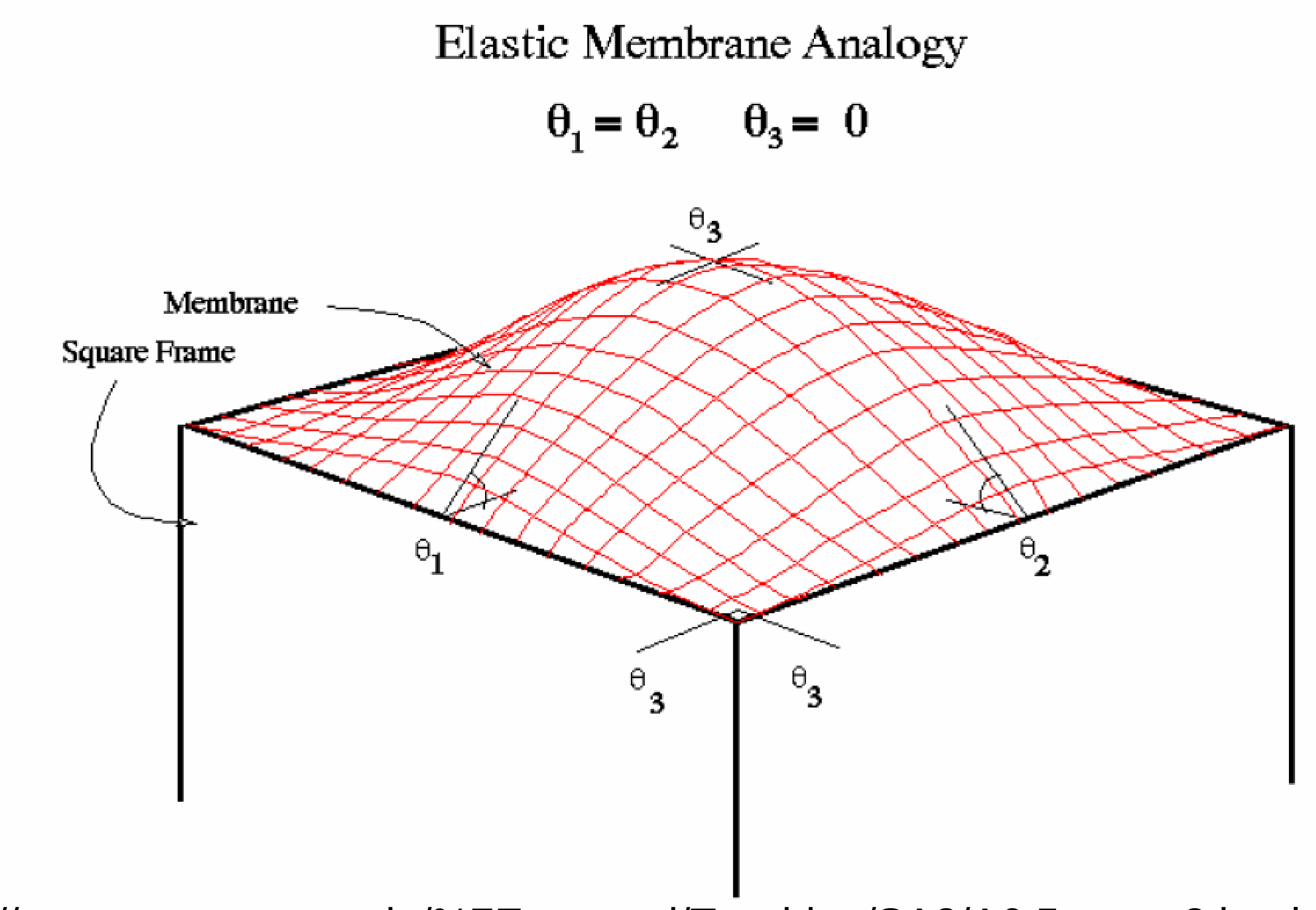
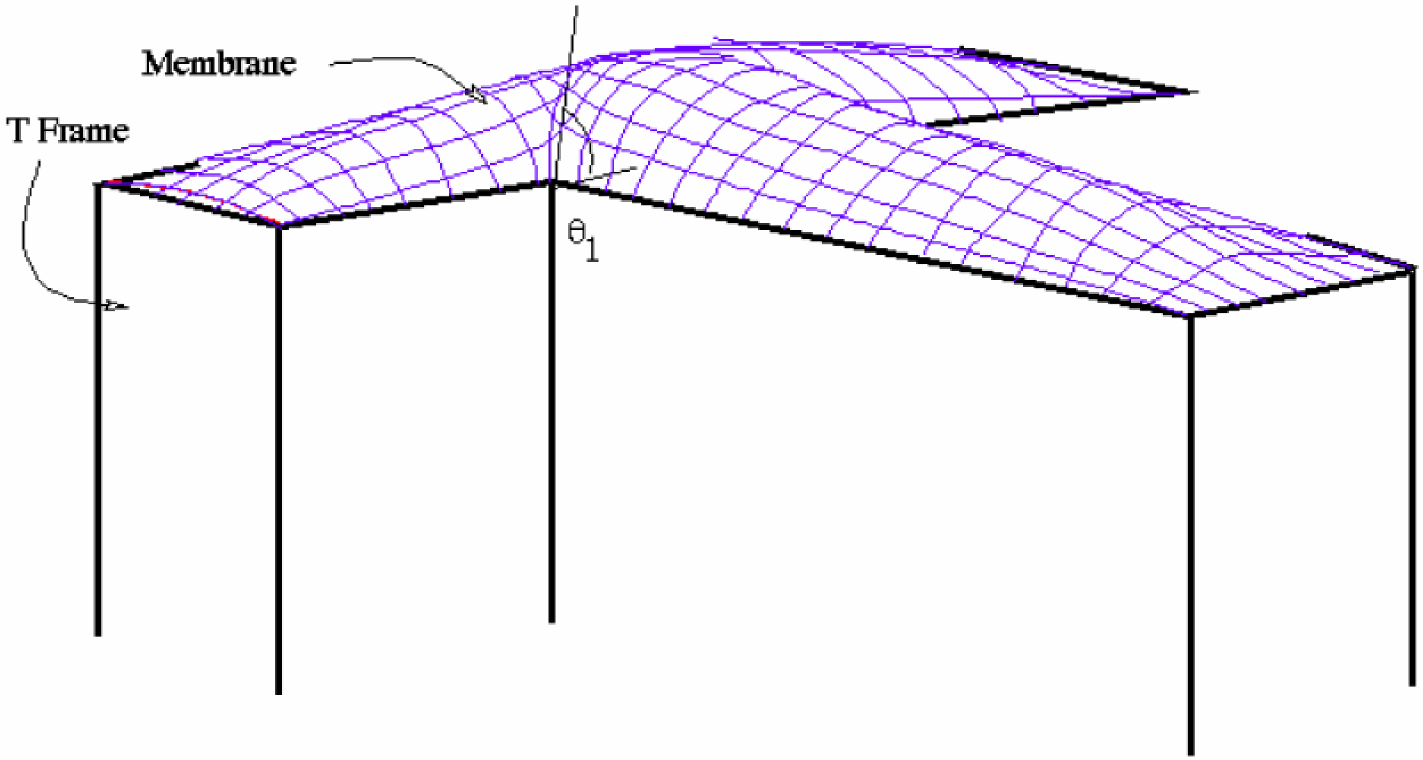


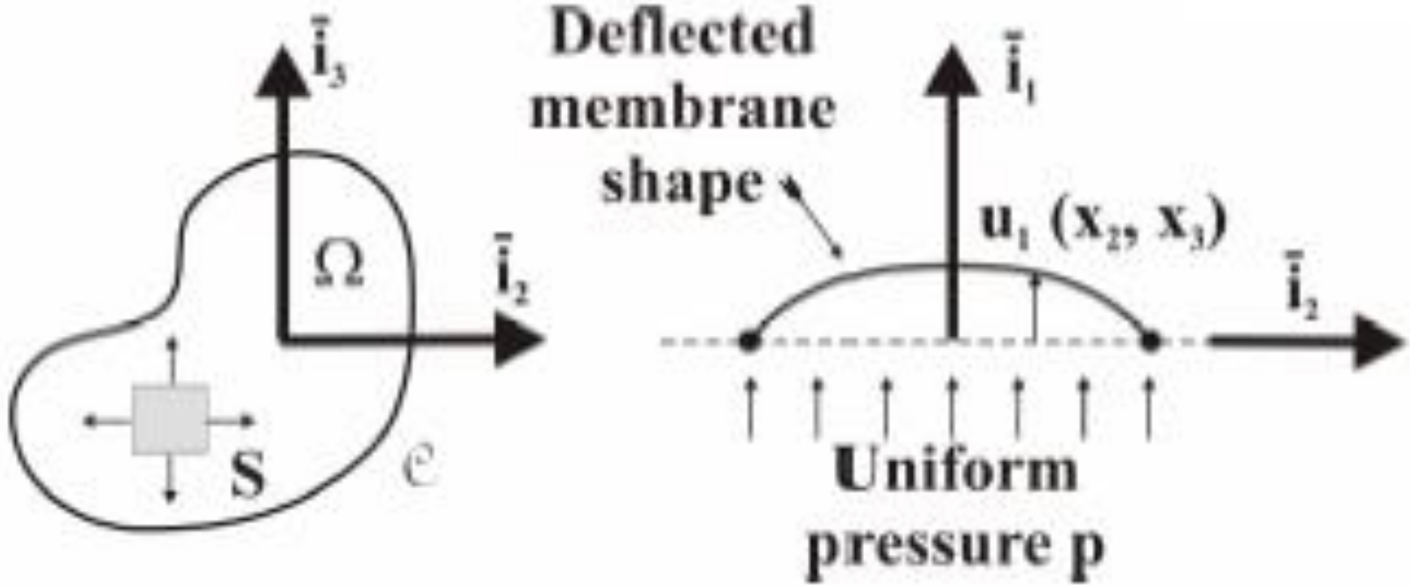
http://www.ae.msstate.edu/%7Emasoud/Teaching/SA2/A6.5_more2.html



http://www.ae.msstate.edu/%7Emasoud/Teaching/SA2/A6.5_more3.html

Elastic Membrane Analogy
θ_1 = Maximum





The thin membrane attached to the contour $\mathcal{C}$.

Bauchau and Craig notes, August 2006

Torsion of Non-circular shafts:

The governing equation for the torsion problem is given by

$$\frac{1}{G}\frac{\partial^2\phi}{\partial x^2}+\frac{1}{G}\frac{\partial^2\phi}{\partial y^2}+2\theta = 0$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = -2G\theta$$

$$\tau_{zx} = \frac{\partial\phi}{\partial y} \qquad \tau_{zy} = -\frac{\partial\phi}{\partial x}$$

On the free boundary $\phi = 0$.

Here ϕ - is a stress function

The shear stresses within the shaft are related to the derivatives of ϕ with respect to x and y.

$$\tau_{zx} = \frac{\partial \phi}{\partial y} \quad \text{and} \quad \tau_{zy} = - \frac{\partial \phi}{\partial x}$$

On the free boundary $\phi = 0$. This is the case of a Poisson's Equation

To derive the weak form multiply the governing equation with a weighting function w(x,y)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2G\theta = 0$$

$$\iint \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2G\theta\right) w(x,y)\,dx\,dy = 0$$

$$\iint \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) w(x,y)\,dx\,dy + \iint 2G\theta\, w(x,y)\,dx\,dy = 0$$

$$\iint\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)w(x,y)\,dxdy + \iint 2G\theta\, w(x,y)\,dxdy = 0$$

$$\iint\frac{\partial^2\phi}{\partial x^2}w(x,y)\,dxdy + \iint\frac{\partial^2\phi}{\partial y^2}w(x,y)\,dxdy + \iint 2G\theta\, w(x,y)\,dxdy = 0$$

$$\oint w(x,y)\frac{\partial\phi}{\partial x}n_x - \iint\frac{\partial\phi}{\partial x}\frac{\partial w}{\partial x}\,dxdy$$

$$+\oint w(x,y)\frac{\partial\phi}{\partial y}n_y - \iint\frac{\partial\phi}{\partial y}\frac{\partial w}{\partial y}\,dxdy + \iint 2G\theta\, w(x,y)\,dxdy = 0$$

where n_x and n_y are the components (direction cosines) of the unit normal vector

As Φ is specified along the boundaries w(x,y) = 0 and the boundary terms vanish. The weak form becomes

$$\iint \frac{\partial \phi}{\partial x}\frac{\partial w}{\partial x}\,dxdy + \iint \frac{\partial \phi}{\partial y}\frac{\partial w}{\partial y}\,dxdy = \iint 2G\theta\, w(x,y)\,dxdy$$

Assuming a CST element and substituting Φ as $N_1\Phi_1 + N_2\Phi_2 + N_3\Phi_3$ and w(x,y) as $\boldsymbol{N_1}$, $\boldsymbol{N_2}$, $\boldsymbol{N_3}$ we get a system of 3 equations in 3 unknowns which can be written as

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}$$

Where

$$K_{ij} = \iint \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} dxdy + \iint \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} dxdy$$

$$f_j = \iint 2G\theta \, N_j dxdy = 0$$

$$K_{ij} = \iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\,dxdy + \iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\,dxdy$$

$$K_{11} = \iint \frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x}\,dxdy + \iint \frac{\partial N_1}{\partial y}\frac{\partial N_1}{\partial y}\,dxdy$$

$$\boxed{N_i\,(x,y) = \frac{1}{2A_e}\,(\alpha_i + \beta_i x + \gamma_i\, y)}$$

$$K_{11} = \frac{1}{4A^2}\iint \beta_1\beta_1 dxdy + \frac{1}{4A^2}\iint \gamma_1\gamma_1 dxdy$$

$$= \frac{1}{4A^2}\beta_1\beta_1\iint dxdy + \frac{1}{4A^2}\gamma_1\gamma_1\iint dxdy$$

$$= \frac{1}{4A^2}\left(\beta_1\beta_1 + \gamma_1\gamma_1\right)A = \frac{1}{4A}\left(\beta_1\beta_1 + \gamma_1\gamma_1\right)$$

$$K_{12} = \frac{1}{4A^2}\iint \beta_1\beta_2 dxdy + \frac{1}{4A^2}\iint \gamma_1\gamma_2 dxdy$$

$$= \frac{1}{4A}\left(\beta_1\beta_2 + \gamma_1\gamma_2\right)$$

$$K_{13} = \frac{1}{4A^2}\iint \beta_1\beta_3 dxdy + \frac{1}{4A^2}\iint \gamma_1\gamma_3 dxdy$$

$$= \frac{1}{4A}\left(\beta_1\beta_3 + \gamma_1\gamma_3\right)$$

$$K_{22} = \frac{1}{4A^2}\iint \beta_2\beta_2 dxdy + \frac{1}{4A^2}\iint \gamma_2\gamma_2 dxdy$$

$$= \frac{1}{4A}\left(\beta_2\beta_2 + \gamma_2\gamma_2\right)$$

$$K_{23} = \frac{1}{4A^2}\iint \beta_2\beta_3 dxdy + \frac{1}{4A^2}\iint \gamma_2\gamma_3 dxdy$$

$$= \frac{1}{4A}\left(\beta_2\beta_3 + \gamma_2\gamma_3\right)$$

$$K_{33} = \frac{1}{4A^2}\iint \beta_3\beta_3 dxdy + \frac{1}{4A^2}\iint \gamma_3\gamma_3 dxdy$$

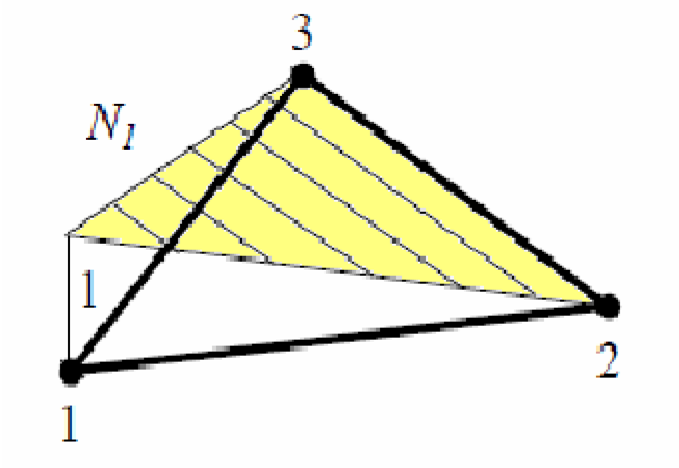
$$= \frac{1}{4A}\left(\beta_3\beta_3 + \gamma_3\gamma_3\right)$$

$$[K] = \frac{1}{4A}\begin{bmatrix} \beta_1^2 + \gamma_1^2 & \beta_1\beta_2 + \gamma_1\gamma_2 & \beta_1\ \beta_3 + \gamma_1\gamma_3 \\ & \beta_1^2 + \gamma_2^2 & \beta_2\beta_3 + \gamma_2\gamma_3 \\ & & \beta_3^2 + \gamma_3^2 \end{bmatrix}$$

$$f_j = \iint 2G\theta \, N_j dxdy = 0$$

$$f_1 = \iint 2G\theta \, N_1 dxdy = 0$$

$$= 2G\theta \frac{A}{3}$$



$$f_2 = \iint 2G\theta \, N_2 dxdy = 0$$

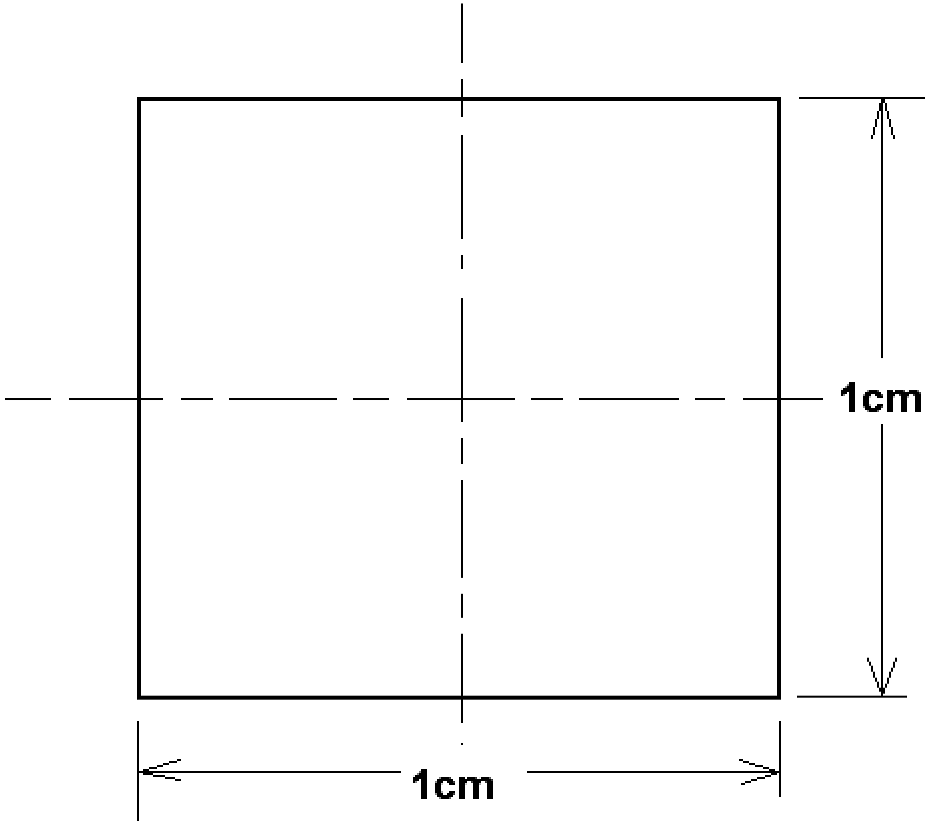
$$= 2G\theta \frac{A}{3}$$

$$f_3 = \iint 2G\theta\, N_3 dxdy = 0$$

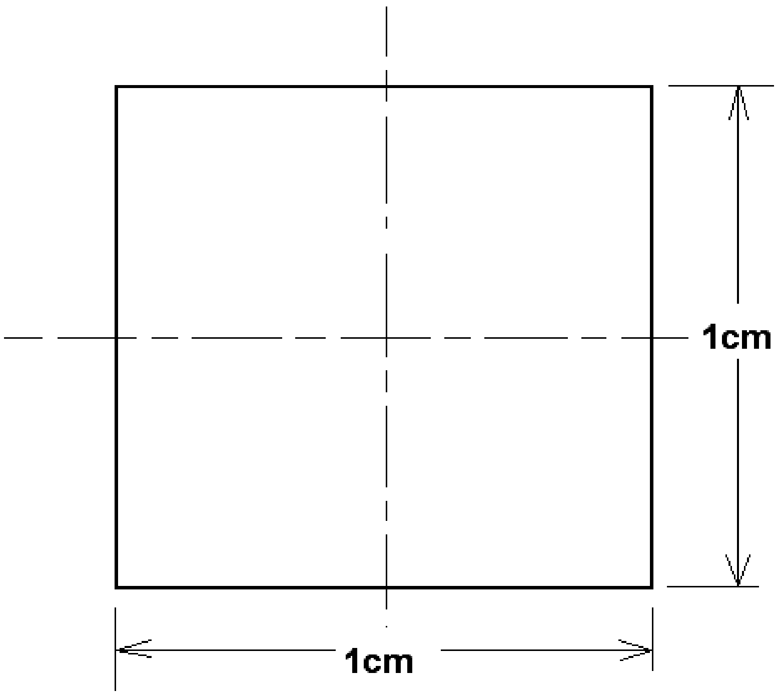
$$= 2G\theta \frac{A}{3}$$

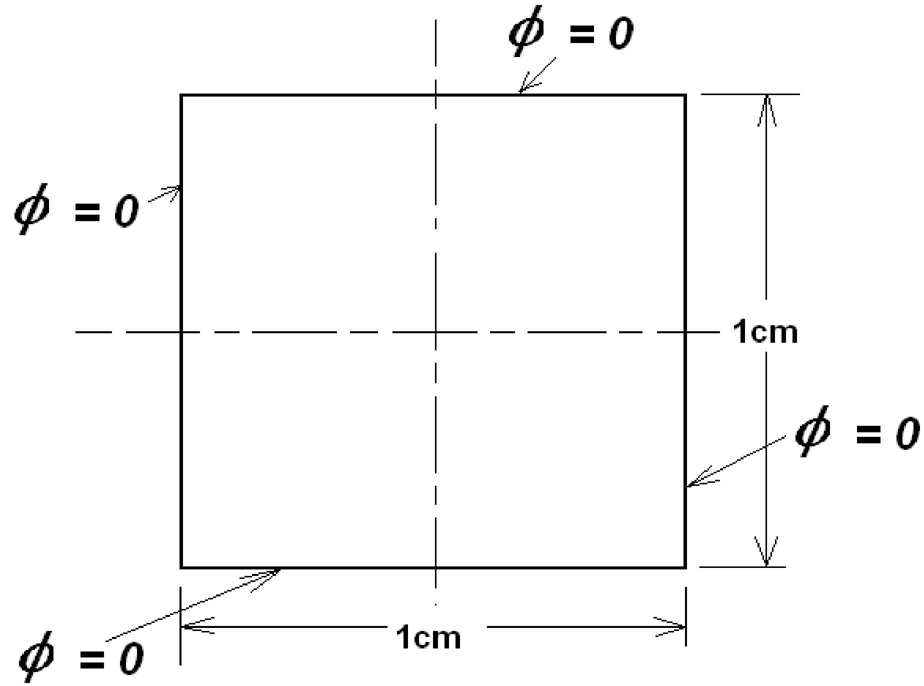
$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = 2G\theta \frac{A}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Problem: Determine the stresses in a shaft of square cross section as shown in fig.

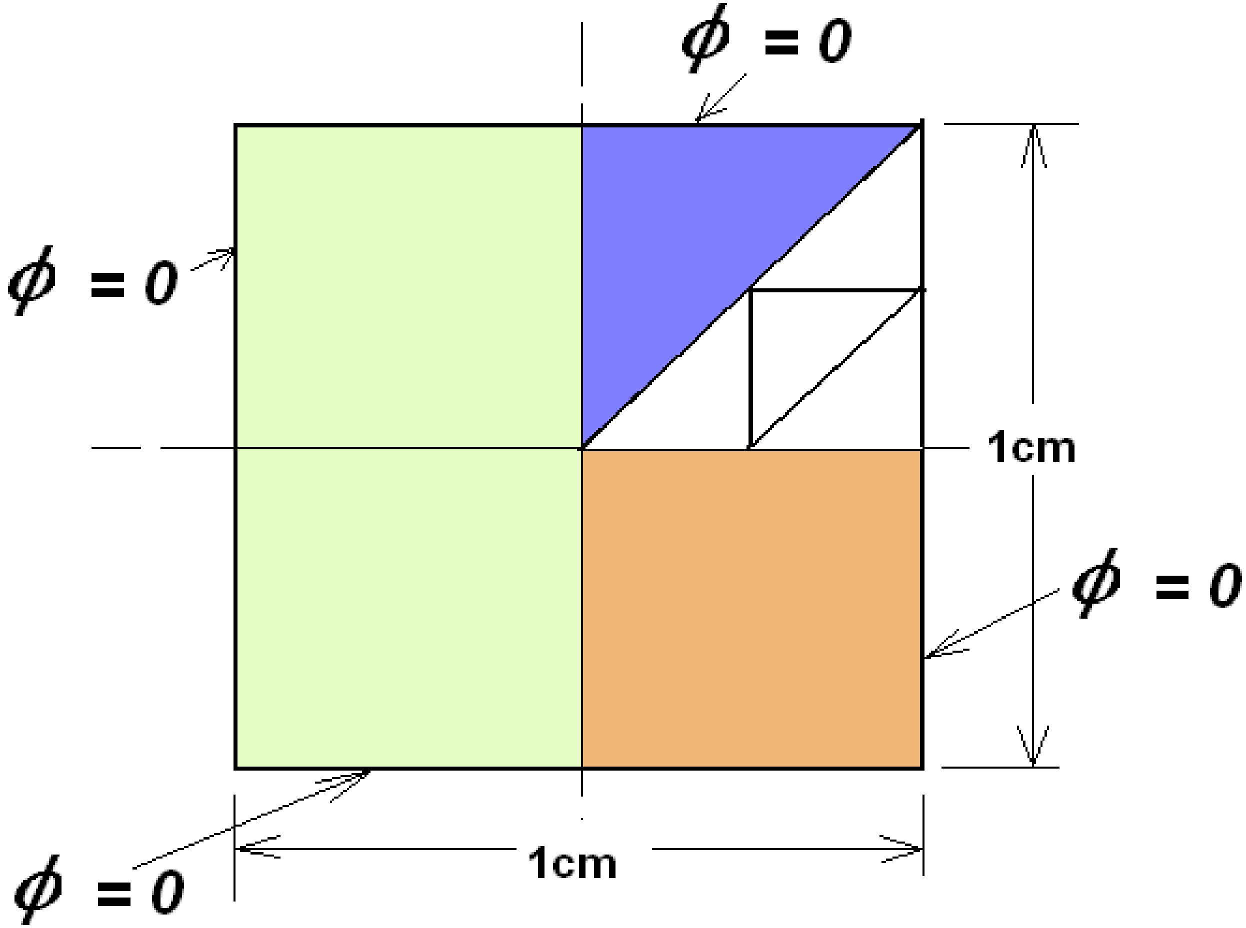


$$2G\theta = 2790$$

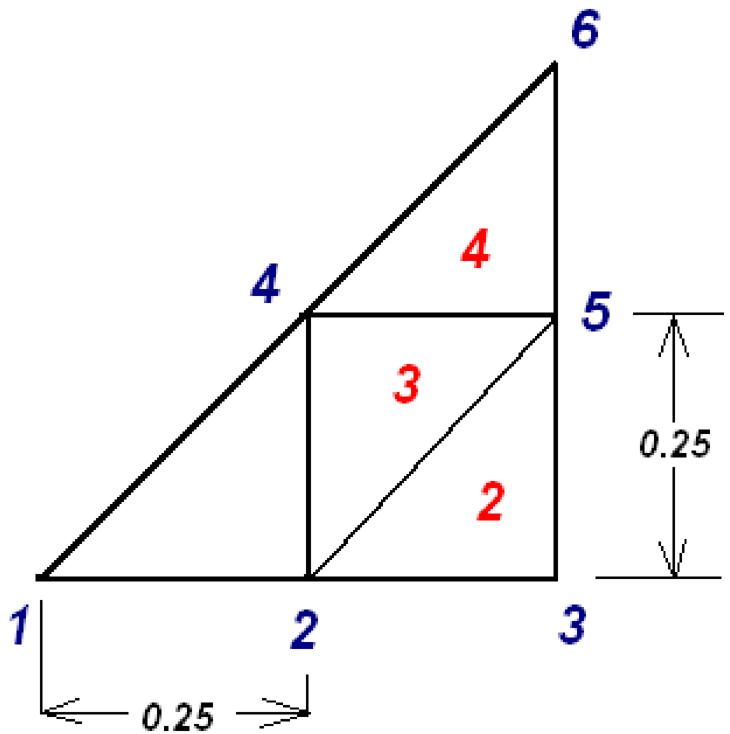




$$2G\theta = 2790$$

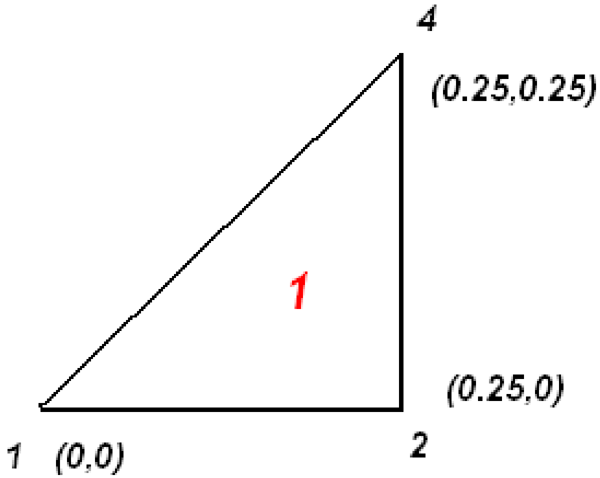


$$2G\theta = 2790$$



Element Connectivity

Element No.	i	j	k
1	1	2	4
2	2	3	5
3	5	4	2
4	4	5	6



$$\text{Area} = \frac{1}{2} xbxht = \frac{1}{2} x \frac{1}{4} x \frac{1}{4} = \frac{1}{32}$$

$\alpha_i = x_j y_k - x_k y_j$	$\beta_i = y_j - y_k$	$\gamma_i = -(x_j - x_k)$
0.0625	-0.25	0
0	0.125	-0.25
0	0	0.25

$$[K]^1 = \frac{1}{4A}\begin{bmatrix} \beta_1^2+\gamma_1^2 & \beta_1\beta_2+\gamma_1\gamma_2 & \beta_1\ \beta_3+\gamma_1\gamma_3 \\ & \beta_1^2+\gamma_2^2 & \beta_2\beta_3+\gamma_2\gamma_3 \\ & & \beta_3^2+\gamma_3^2 \end{bmatrix}$$

$$= 8\begin{bmatrix} 0.0625 & -0.0625 & 0 \\ -0.0625 & 0.125 & -0.0625 \\ 0 & -0.0625 & 0.0625 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$
[K]^1 = \frac{1}{2}\;
\begin{array}{c}
 \\ 1 \\ 2 \\ 4
\end{array}
\begin{array}{c}
\begin{array}{ccc} 1 & 2 & 4 \end{array} \\
\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}
\end{array}
$$

	1	2	3	4	5	6
1	1	-1		0		
2	-1	2		-1		
3						
4	0	-1		1		
5						
6						

$$[K]^2 = \frac{1}{2} \begin{array}{c c} & \begin{array}{ccc} 2 & 3 & 5 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

	1	2	3	4	5	6
1	1	-1		0		
2	-1	2+1	-1	-1	0	
3		-1	2		-1	
4	0	-1		1		
5		0	-1		1	
6						

$$[K]^3 = \frac{1}{2}\begin{array}{c|ccc} & 5 & 4 & 2 \\ \hline 5 & 1 & -1 & 0 \\ 4 & -1 & 2 & -1 \\ 2 & 0 & -1 & 1 \end{array}$$

	1	2	3	4	5	6
1	1	-1		0		
2	-1	2+1+1	-1	-1-1	0+0	
3		-1	2		-1	
4	0	-1-1		1+2	-1	
5		0+0	-1	-1	1+1	
6						

$$[K]^4 = \frac{1}{2}\begin{array}{c c} & \begin{array}{ccc} 4 & 5 & 6 \end{array} \\ \begin{array}{c} 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

	1	2	3	4	5	6
1	1	-1		0		
2	-1	2+1 +1	-1	-1-1	0+0	
3		-1	2		-1	
4	0	-1-1		1+2+1	-1-1	0
5		0+0	-1	-1-1	1+1 +2	-1
6				0	-1	1

$$[k] = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & -2 & 0 & 0 \\ 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 4 & -2 & 0 \\ 0 & 0 & -1 & -2 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Semi bandwith =(Max. diff. bet node nos +1)x DOF

= (3+1)x1=4

$$2G\theta = 2790\ N/mm^2$$

$$\{f\}^e = \frac{2G\theta \times A}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{2790}{3\times 32}\begin{bmatrix}1\\1\\1\end{bmatrix} = 29.06\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= 29.06 \begin{Bmatrix}1\\1\\1\end{Bmatrix} \begin{matrix}1\\2\\4\end{matrix}$$

	1
1	1
2	1
3	
4	1
5	
6	

$$= 29.06 \begin{Bmatrix}1\\1\\1\end{Bmatrix} \begin{matrix}2\\3\\5\end{matrix}$$

	2
1	1
2	1+ 1
3	1
4	1
5	1
6	

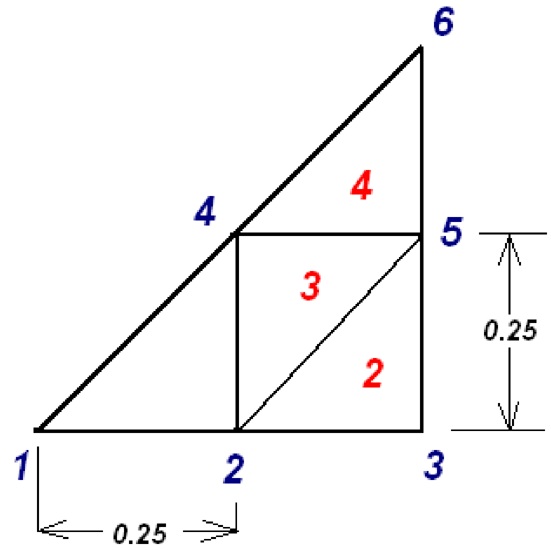
$$= 29.06 \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \quad \begin{matrix} 5 \\ 4 \\ 2 \end{matrix}$$

	1
1	1
2	1+1+1
3	1
4	1+1
5	1+1
6	

$$= 29.06 \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \quad \begin{matrix} 4 \\ 5 \\ 6 \end{matrix}$$

	1
1	1
2	1+1+1
3	1
4	1+1+1
5	1+1+1
6	1

$$[k]\begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\end{bmatrix} = [$$



$$f = 29.06\begin{bmatrix}1\\3\\1\\3\\3\\1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & -1 & 0 & 0 & 0 & 0\\ -1 & 4 & -1 & -2 & 0 & 0\\ 0 & -1 & 2 & 0 & -1 & 0\\ 0 & -2 & 0 & 4 & -2 & 0\\ 0 & 0 & -1 & -2 & 4 & -1\\ 0 & 0 & 0 & 0 & -1 & 1\end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\end{bmatrix} = 29.06\begin{bmatrix}1\\3\\1\\3\\3\\1\end{bmatrix}$$

Solving we get

$\phi_1 = 217.95$

$\phi_2 = 159.83$

$\phi_4 = 123.505$

$$\tau_{xz} = \frac{\partial \phi}{\partial y}\ , \ \tau_{yz} = -\frac{\partial \phi}{\partial x}$$

$$\tau_{xz} = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\{N_1\phi_1 + N_2\phi_2 + N_3\phi_4\}$$

$$= \{\gamma_1\phi_1 + \gamma_2\phi_2 + \gamma_3\phi_4\}, \ = 16\,(-9.08125)$$

$= -144\ \mathrm{N/mm^2}$

$$\tau_{xz} = -\frac{\partial \phi}{\partial x} = -\frac{\partial}{\partial x}\{N_1\phi_1 + N_2\phi_2 + N_3\phi_4\}$$

$$= \{\boldsymbol{\beta}_1\phi_1 + \boldsymbol{\beta}_2\phi_2 + \boldsymbol{\beta}_3\phi_4\}$$

$$= -16\ (-14.53) = 232.48\ \text{N/mm}^2$$

For element 2

$$\tau_{xz} = \frac{1}{2A}\{\gamma_1\phi_2 + \gamma_2\phi_3 + \gamma_3\phi_5\}$$

$$\tau_{yz} = -\frac{1}{2A}\{\boldsymbol{\beta}_1\phi_2 + \boldsymbol{\beta}_2\phi_3 + \boldsymbol{\beta}_3\phi_5\}$$

For element 3

$$\tau_{xz} = \{\gamma_1\phi_5+\gamma_2\phi_4+\gamma_3\phi_2\}$$

$$\tau_{yz} = -\{\boldsymbol{\beta}_1\phi_5 + \boldsymbol{\beta}_2\phi_4 + \boldsymbol{\beta}_3\phi_2\}$$

For element 4

$$\tau_{xz} = \{\gamma_1\phi_4+\gamma_2\phi_5+\gamma_3\phi_6\}$$

$$\tau_{yz} = -\{\boldsymbol{\beta}_1\phi_4 + \boldsymbol{\beta}_2\phi_5 + \boldsymbol{\beta}_3\phi_6\}$$

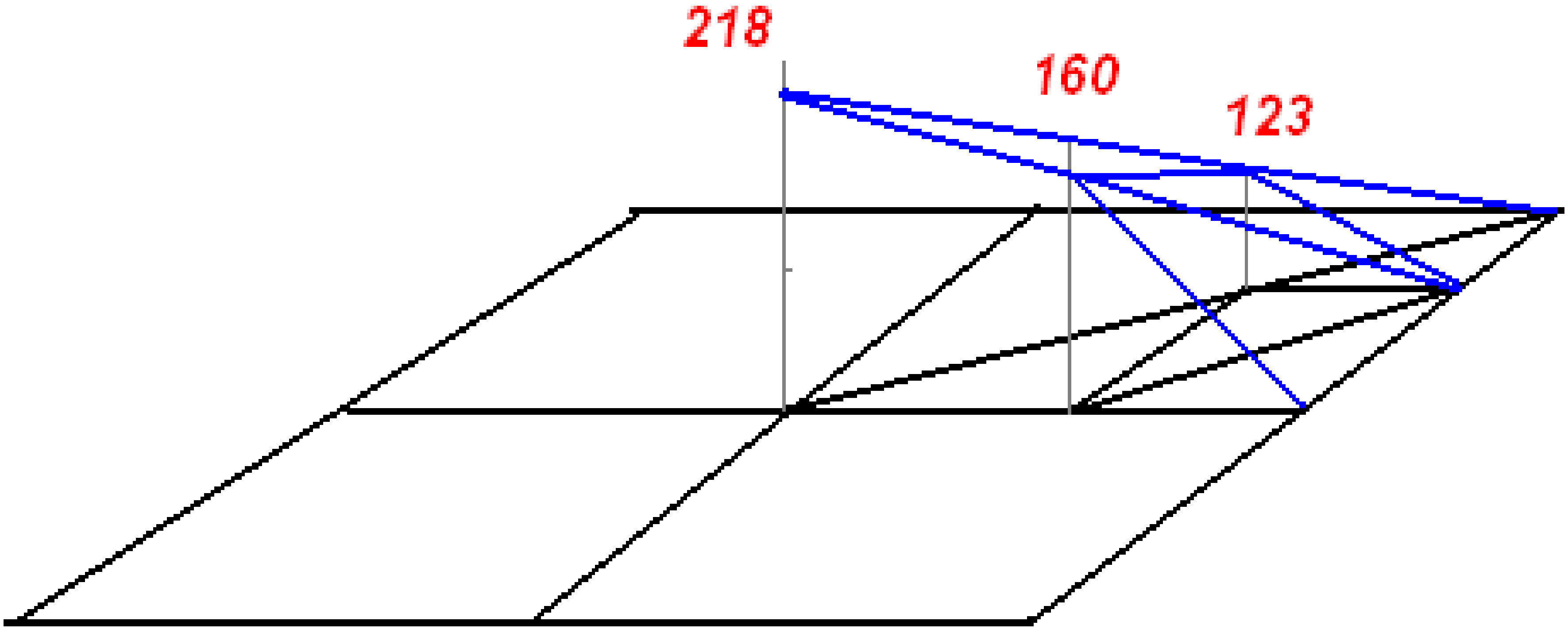
$$T^1 = 2\int \phi dA$$

$$= 2\int (N_1\phi_1 + N_2\phi_2 + N_3\phi_4)dA$$

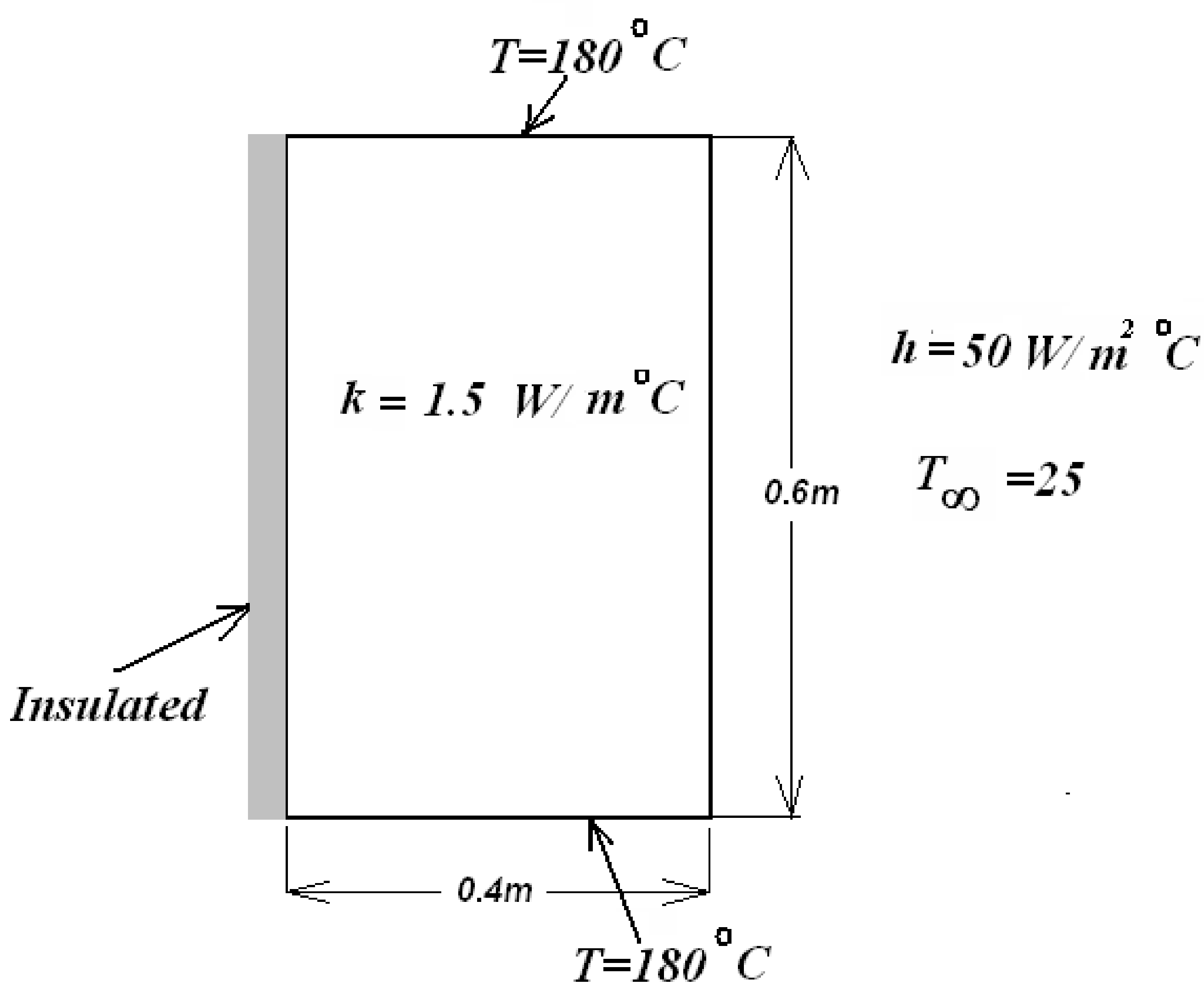
$$= \frac{2}{3}\{\phi_1 + \phi_2 + \phi_4\}.A$$

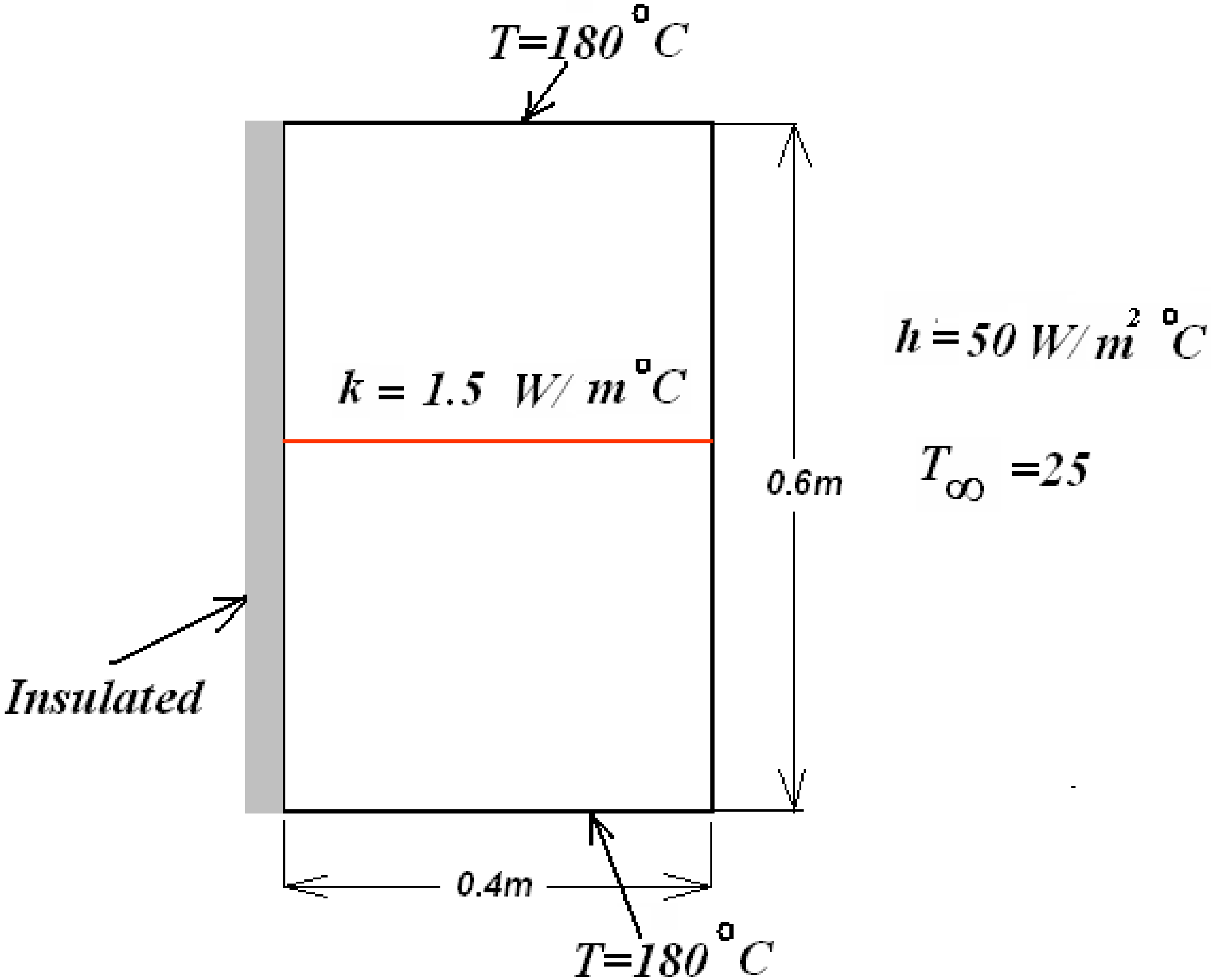
$Similarly\ determine \quad T^2\ T^3\ \&T^4$

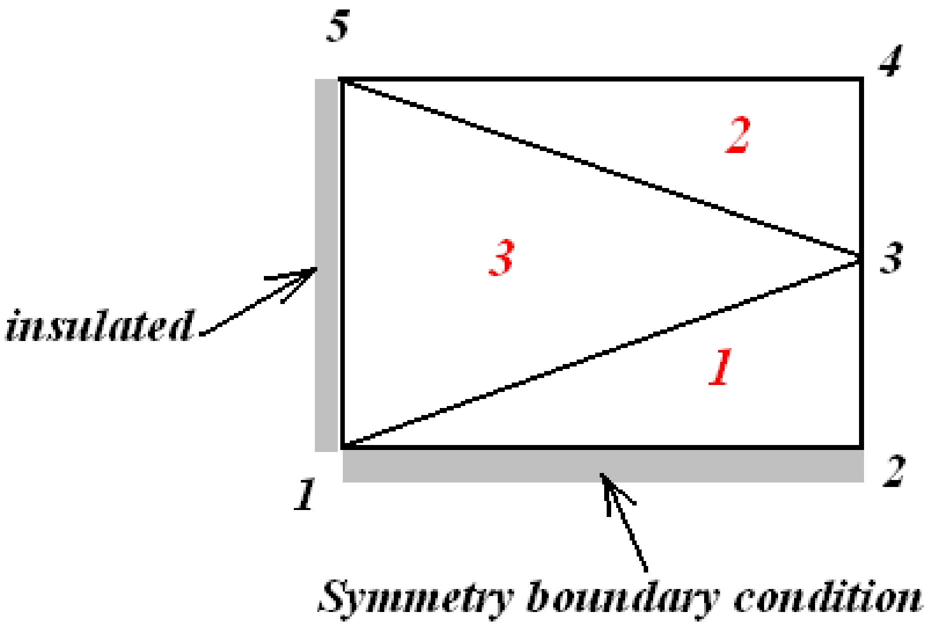
$Total\ Torque = \quad \left(T^1 + T^2 + T^3 + T^4\right) * 8$


218
160
123

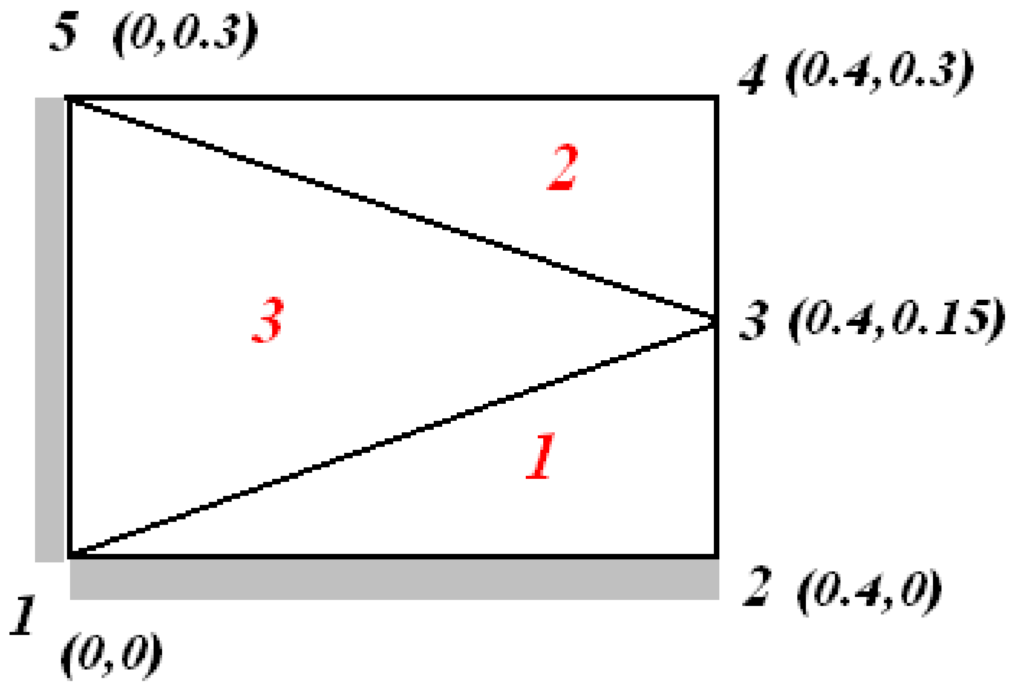
PROBLEM 2:




T=180 °C
k = 1.5 W/ m°C
h = 50 W/ m² °C
0.6m
T∞ =25
Insulated
0.4m
T=180 °C



Element	i	j	k
1	1	2	3
2	5	4	3
3	1	3	5


5 (0,0.3)
4 (0.4,0.3)
2
3
3 (0.4,0.15)
1
1
(0,0)
2 (0.4,0)

$$k\left\{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right\}=Q$$

Element	i	j	k
1	1	2	3
2	5	4	3
3	1	3	5

$$[k]^e = \frac{k}{4A}\begin{bmatrix} \beta_1^2+\gamma_1^2 & \beta_1\beta_2+\gamma_1\gamma_2 & \beta_1\ \beta_3+\gamma_1\gamma_3 \\ \beta_2\beta_1+\gamma_1\gamma_2 & \beta_1^2+\gamma_2^2 & \beta_2\beta_3+\gamma_2\gamma_3 \\ \beta_1\beta_3+\gamma_1\gamma_3 & \beta_2\beta_3+\gamma_2\gamma_3 & \beta_3^2+\gamma_3^2 \end{bmatrix}$$

Element 1 and 2	Element 3
$\beta_1 = -0.15$, $\gamma_1 = 0$,	$\beta_1 = 0.15$, $\gamma_1 = 0$,
$\beta2 = 0.15$, $\gamma_2 = -0.4$	$\beta_2 = 0.15$, $\gamma_2 = -0.4$
$\beta3 = 0$, $\gamma_3 = 0.4$	$\beta_3 = 0$, $\gamma_3 = 0.4$

$\beta_1 = (-0.15)\ (-1)$, $\beta_2 = 0.3$, $\beta_3 = -\ 0.15$

$\gamma_1 = -0.4$, $\gamma_2 = 0$, $\gamma_3 = 0.4$

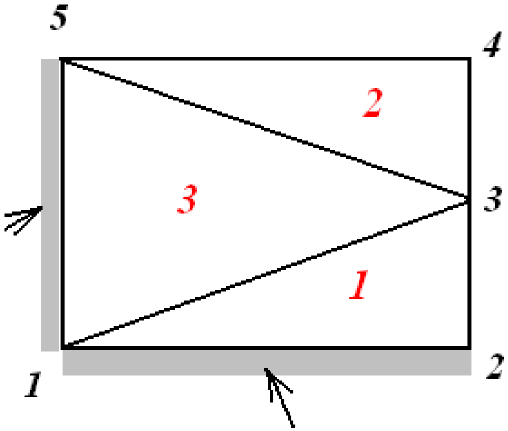
$$[k]^2_{cond} = [k]^1_{cond} = \frac{1.5}{\frac{4}{2} \times 0.4 \times 0.15} \begin{bmatrix} 0.0225 & -0.0225 & 0 \\ -0.0225 & 0.1825 & -0.16 \\ 0 & -0.16 & 0.16 \end{bmatrix}$$

$$= 10 \begin{bmatrix} 0.028125 & -0.028125 & 0 \\ -0.028125 & 0.228125 & -0.2 \\ 0 & -0.2 & 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.28125 & -0.28125 & 0 \\ -0.28125 & 2.28125 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$$[k]^3_{cond} = \frac{1.5}{4\times 2\times \frac{1}{2}\times 0.4\times 0.15}\begin{bmatrix} 0.1825 & -0.045 & -0.1825 \\ -0.045 & 0.09 & -0.045 \\ -0.1825 & -0.045 & 0.1825 \end{bmatrix}$$

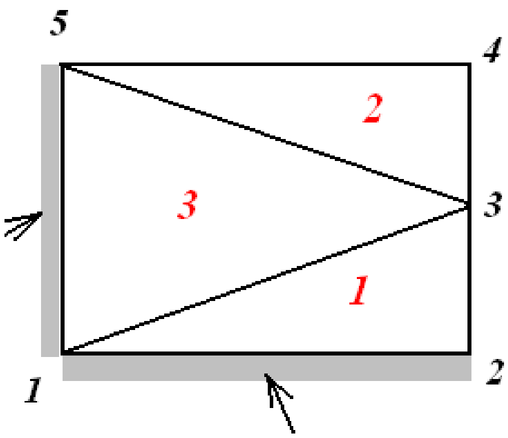
$$= \begin{bmatrix} 1.14 & -0.28125 & -0.86 \\ & 0.5625 & -0.28125 \\ & & 1.14 \end{bmatrix}$$



$$[k]_{conv} = \frac{hpl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow p = 1$$

$$[k]^2_{conv} = [k]^1_{conv} = \frac{hl}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$



$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2.5 & 1.25 \\ 0 & 1.25 & 2.5 \end{bmatrix}$$

$$Q = \frac{hlT_\infty}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 93.75 \\ 93.75 \end{Bmatrix}$$

$$[k]^{e}{}_{Thermal} = [k]_{condn} + [k]_{conv}$$

$$[k]_e^1 = [k]_e^2 = \begin{bmatrix} 0.28125 & -0.28125 & 0 \\ -0.28125 & 4.78 & -0.75 \\ 0 & -0.75 & 4.5 \end{bmatrix}$$

$$[[k]_e^3 = \begin{bmatrix} 1.14 & -0.28125 & -0.86 \\ -0.28125 & 0.5625 & -0.28125 \\ -0.86 & -0.28125 & 1.14 \end{bmatrix}$$

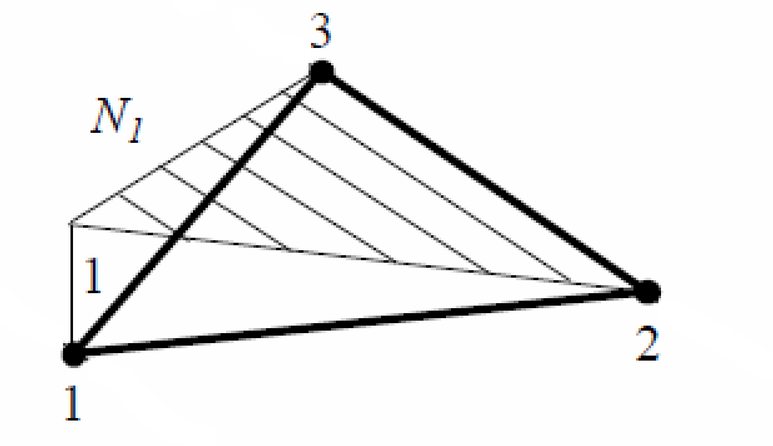
$$[k]_{et}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = [Q]^G \quad \Rightarrow \quad [Q]^G = \begin{Bmatrix} 0 \\ 93.75 \\ 93.75 + 93.75 \\ 93.75 \\ 0 \end{Bmatrix}$$

$$[k]^G = \begin{bmatrix} 1.42125 & -0.28125 & -0.28125 & 0 & -0.86 \\ -0.28125 & 4.78 & -0.75 & 0 & 0 \\ -0.28125 & -0.75 & 9.5625 & -0.75 & -0.28125 \\ 0 & 0 & -0.75 & 4.78 & -0.28125 \\ -0.86 & 0 & -0.28125 & -0.28125 & 1.42125 \end{bmatrix}$$

$$\begin{bmatrix} 1.42125 & -0.28125 & -0.28125 & 0 & -0.86 \\ -0.28125 & 4.78 & -0.75 & 0 & 0 \\ -0.28125 & -0.75 & 9.5625 & -0.75 & -0.28125 \\ 0 & 0 & -0.75 & 4.78 & -0.28125 \\ -0.86 & 0 & -0.28125 & -0.28125 & 1.42125 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = 93.75 \begin{Bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1.42125 & -0.28125 & -0.28125 & 0 & -0.86 \\ -0.28125 & 4.78 & -0.75 & 0 & 0 \\ -0.28125 & -0.75 & 9.5625 & -0.75 & -0.28125 \\ 0 & 0 & -0.75 & 4.78 & -0.28125 \\ -0.86 & 0 & -0.28125 & -0.28125 & 1.42125 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = 93.75 \begin{Bmatrix} 0 + 0.86 * 180 \\ 1 \\ 2 + 0.28125 * 180 \\ 1 + 0.28125 * 180 \\ 0 \end{Bmatrix}$$

Substitute for T_5 as 80º and evaluate T_1, T_2 ,T_3 and T_4

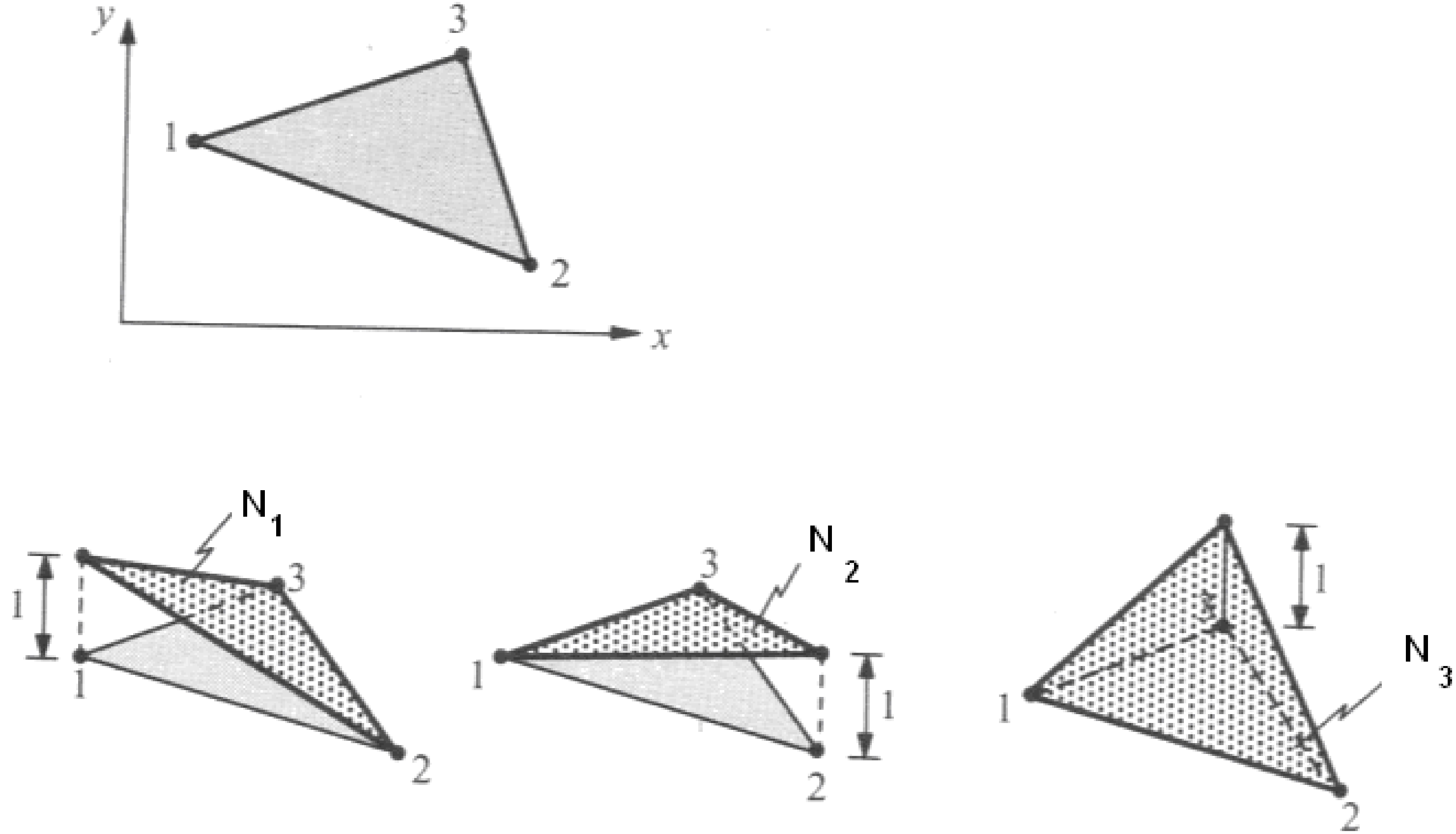


Shape Function N_1 for CST

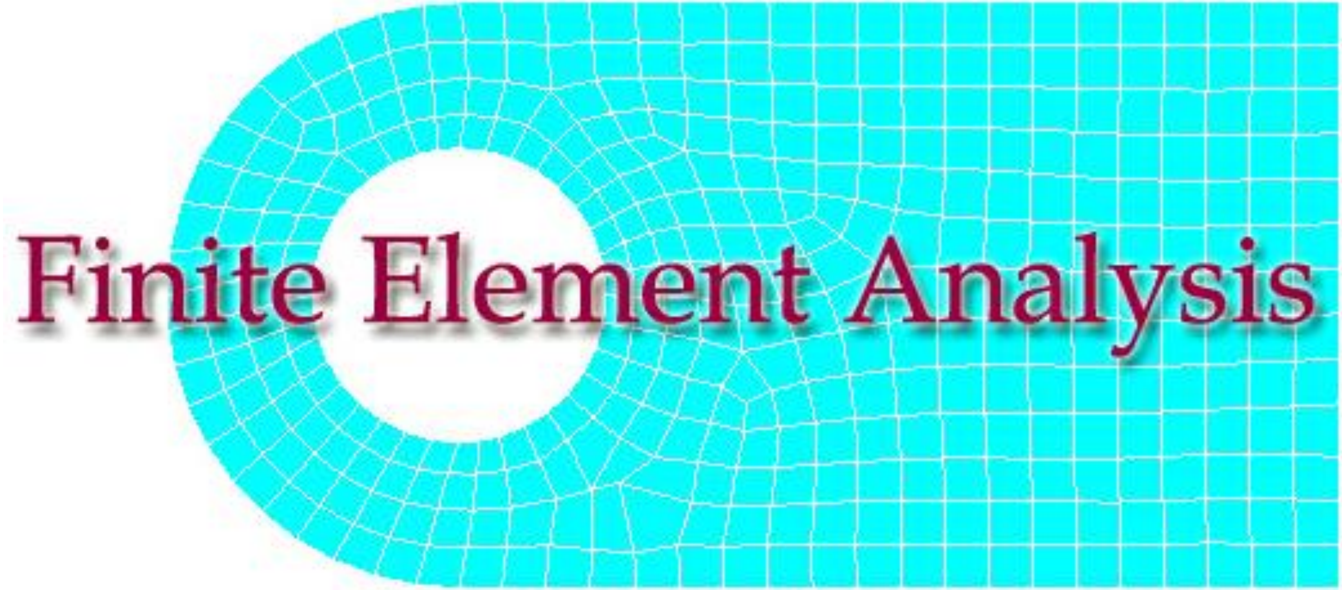
$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{\mathrm{yy}} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \varepsilon_{\mathrm{xy}} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_{xx} = \frac{1}{2A}(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3)$$

$$\varepsilon_{\mathrm{yy}} = \frac{1}{2A}(\gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3)$$



Variation of Shape functions for CST element



TWO DIMENSIONAL ELEMENTS- THERMAL PROBLEMS

LECTURE 9

Types of 2D Problems

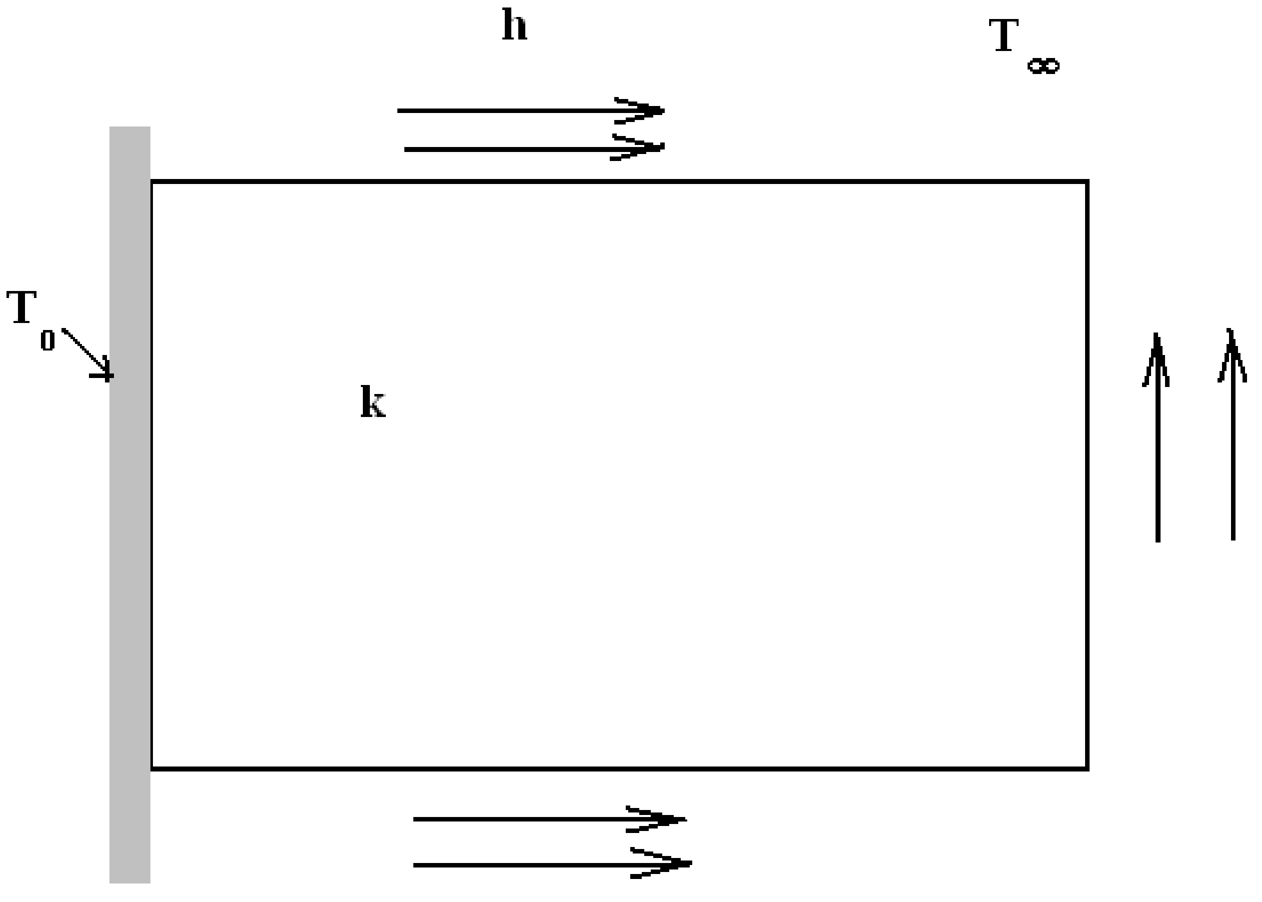
- **VECTOR VARIABLE PROBLEMS**

  e.g. Structural problems

- **SCALAR VARIABLE PROBLEMS**

  e.g. Torsion of non-circular shafts,

  Heat transfer through fins


h
T_∞
T_0
k

Governing Equation for 2D Heat transfer by conduction and convection

$$k\left\{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right\}-h(T-T_\infty)=0$$

Weak form of the equation

$$\iint \frac{\partial T}{\partial x}\frac{\partial w}{\partial x}dxdy+\iint \frac{\partial T}{\partial y}\frac{\partial w}{\partial y}dxdy+\iint hTw(x,y)dxdy$$
$$=\iint hT_\infty\; w(x,y)dxdy$$

$$K_{ij_{condn.}} = k\left[\iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}dxdy + \iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}dxdy\right]$$

$$\boxed{N_i\ (x,y) = \frac{1}{2A_e}\ (\alpha_i + \beta_i x + \gamma_i\ y)}$$

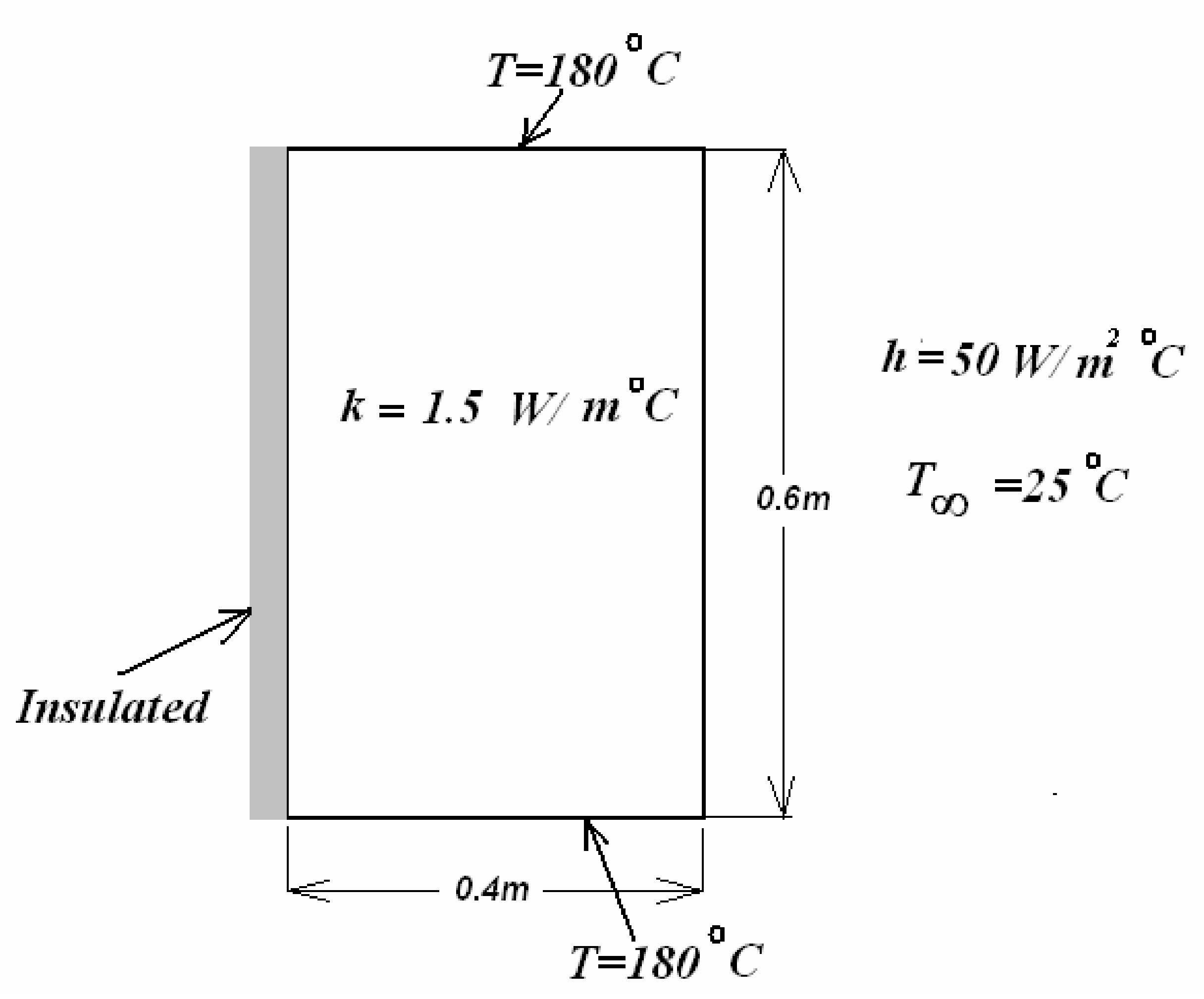
$$[K]^e{}_{condn.} = \frac{k}{4A}\begin{bmatrix} \beta_1^2 + \gamma_1^2 & \beta_1\beta_2 + \gamma_1\gamma_2 & \beta_1\ \beta_3 + \gamma_1\gamma_3 \\ \beta_1\beta_2 + \gamma_1\gamma_2 & \beta_2^2 + \gamma_2^2 & \beta_2\beta_3 + \gamma_2\gamma_3 \\ \beta_1\ \beta_3 + \gamma_1\gamma_3 & \beta_2\beta_3 + \gamma_2\gamma_3 & \beta_3^2 + \gamma_3^2 \end{bmatrix}$$

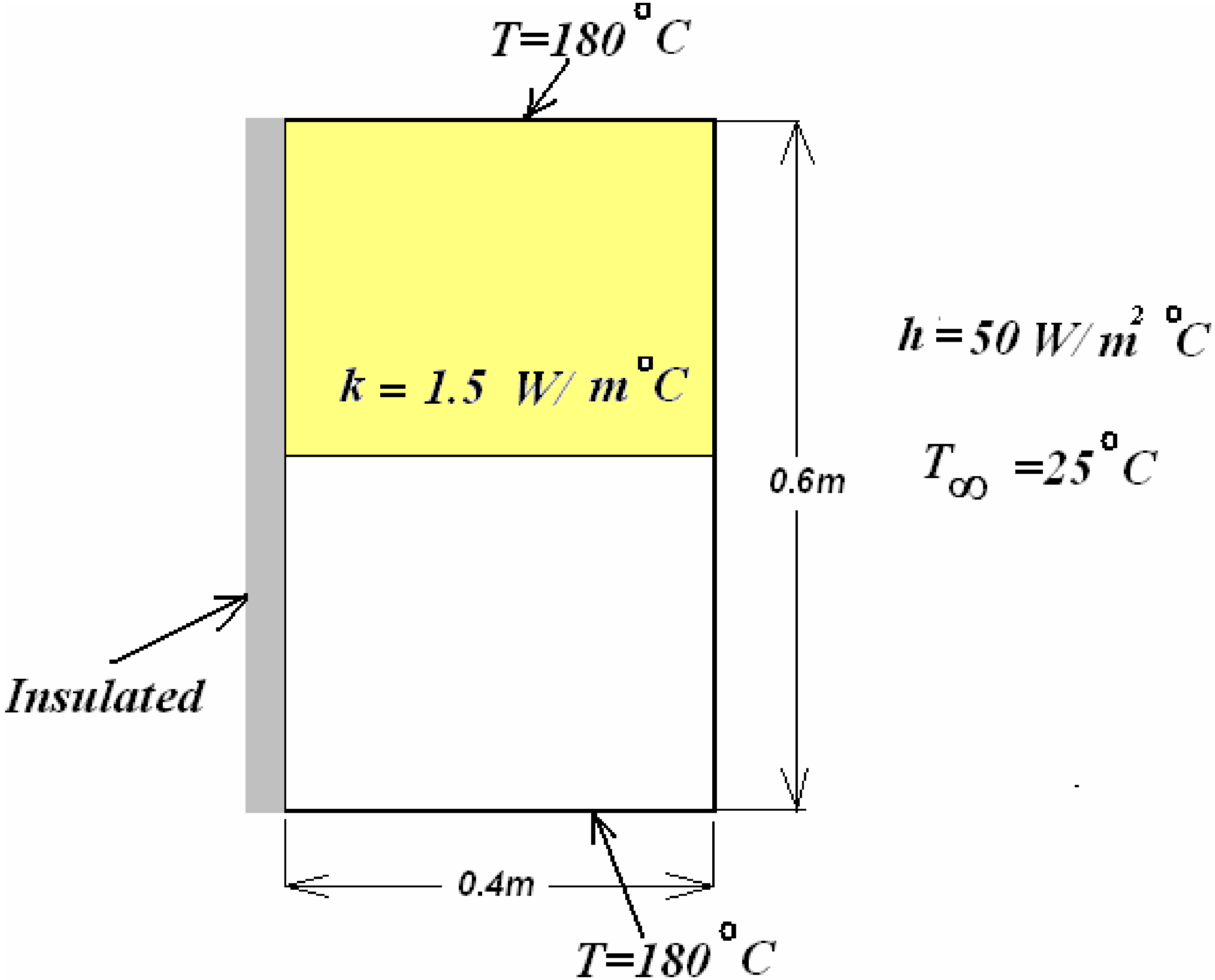
$$[k]_{conv} = \frac{hpl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow p = 1$$

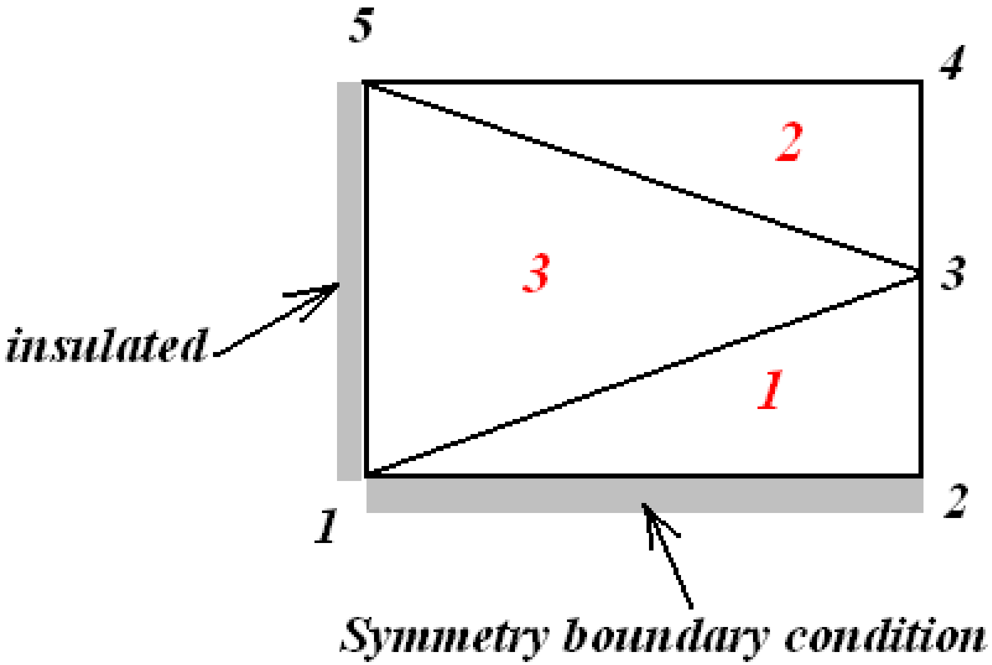
$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \frac{hlT_\infty}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

PROBLEM 1:

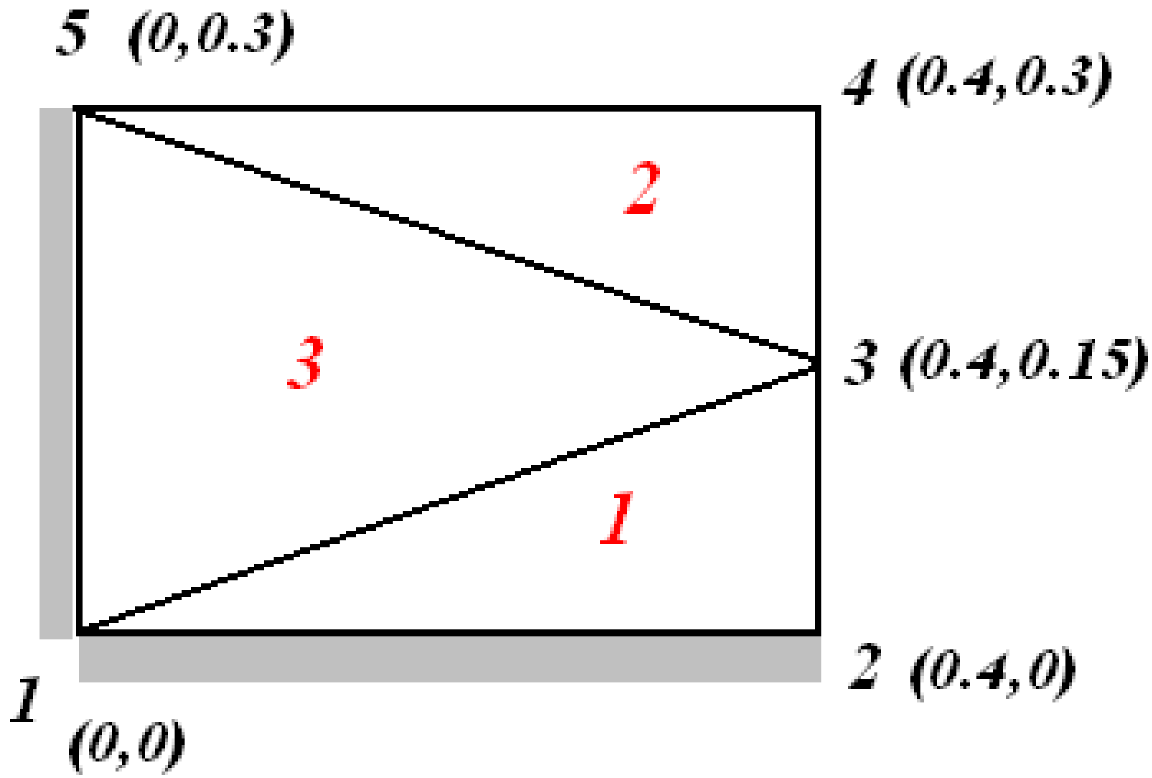




T=180 °C
k = 1.5 W/ m °C
h = 50 W/ m² °C
T∞ =25 °C
0.6m
Insulated
0.4m
T=180 °C



Element	i	j	k
1	1	2	3
2	5	4	3
3	1	3	5


5 (0,0.3)
4 (0.4,0.3)
2
3
3 (0.4,0.15)
1
1
(0,0)
2 (0.4,0)

$$[k]^e = \frac{k}{4A}\begin{bmatrix} \beta_1^2 + \gamma_1^2 & \beta_1\beta_2 + \gamma_1\gamma_2 & \beta_1\ \beta_3 + \gamma_1\gamma_3 \\ \beta_2\beta_1 + \gamma_1\gamma_2 & \beta_2^2 + \gamma_2^2 & \beta_2\beta_3 + \gamma_2\gamma_3 \\ \beta_1\beta_3 + \gamma_1\gamma_3 & \beta_2\beta_3 + \gamma_2\gamma_3 & \beta_3^2 + \gamma_3^2 \end{bmatrix}$$

Element 1 and 2		Element 3	
$\beta_1 = -0.15$	, $\gamma_1 = 0$,	$\beta_1 = 0.15$	, $\gamma_1 = -0.4$
β2 = 0.15	, $\gamma_2 = -0.4$	$\beta_2 = 0.3$	, $\gamma_2 = 0$
β3 = 0	, $\gamma_3 = 0.4$	$\beta_3 = -0.15$	, $\gamma_3 = 0.4$

$$[k]_{cond}^{2} = [k]_{cond}^{1} = \frac{1.5}{\frac{4}{2}\times 0.4\times 0.15}\begin{bmatrix} 0.0225 & -0.0225 & 0 \\ -0.0225 & 0.1825 & -0.16 \\ 0 & -0.16 & 0.16 \end{bmatrix}$$

$$= 10\begin{bmatrix} 0.028125 & -0.028125 & 0 \\ -0.028125 & 0.228125 & -0.2 \\ 0 & -0.2 & 0.2 \end{bmatrix}$$

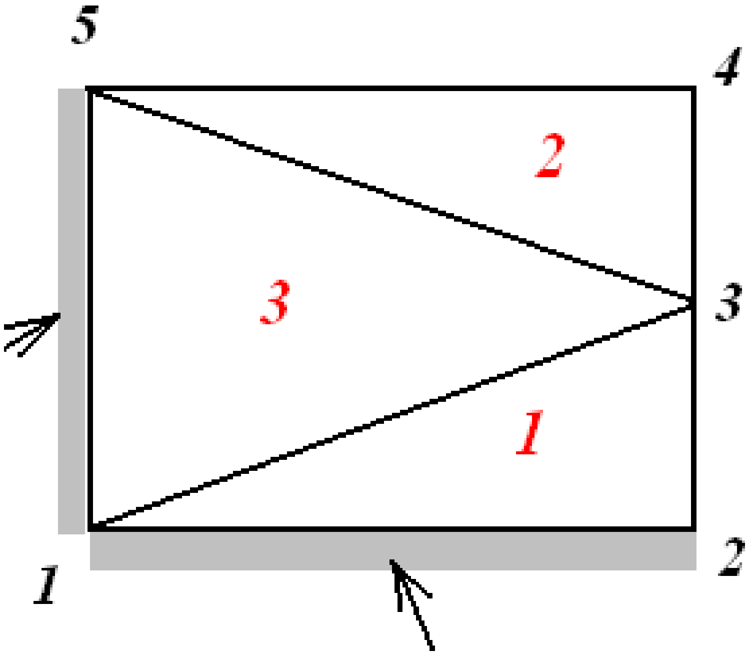
$$= \begin{bmatrix} 0.28125 & -0.28125 & 0 \\ -0.28125 & 2.28125 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$$[k]_{cond}^{3} = \frac{1.5}{4 \times 2 \times \frac{1}{2} \times 0.4 \times 0.15} \begin{bmatrix} 0.1825 & -0.045 & -0.1825 \\ -0.045 & 0.09 & -0.045 \\ -0.1825 & -0.045 & 0.1825 \end{bmatrix}$$

$$= \begin{bmatrix} 1.14 & -0.28125 & -0.86 \\ & 0.5625 & -0.28125 \\ & & 1.14 \end{bmatrix}$$

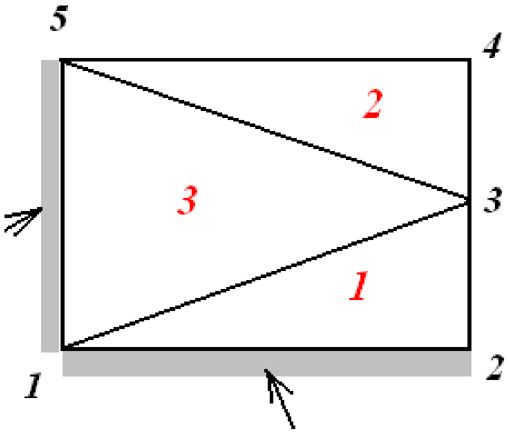
$$[k]_{conv} = \frac{hpl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow \; p = 1$$



Element	i	j	k
1	1	2	3
2	5	4	3
3	1	3	5

$$[k]^{2}_{conv} = [k]^{1}_{conv} = \frac{hl}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$



$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2.5 & 1.25 \\ 0 & 1.25 & 2.5 \end{bmatrix}$$

$$Q = \frac{hlT_{\infty}}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 93.75 \\ 93.75 \end{Bmatrix}$$

$$[k]^{e}_{Thermal} = [k]_{condn} + [k]_{conv}$$

$$[k]^1_{th} = [k]^2_{th} = \begin{bmatrix} 0.28125 & -0.28125 & 0 \\ -0.28125 & 4.78 & -0.75 \\ 0 & -0.75 & 4.5 \end{bmatrix}$$

$$[k]^3_{th} = \begin{bmatrix} 1.14 & -0.28125 & -0.86 \\ -0.28125 & 0.5625 & -0.28125 \\ -0.86 & -0.28125 & 1.14 \end{bmatrix}$$

$$[k]_{th}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = [Q]^G \quad \Rightarrow \quad [Q]^G = \begin{Bmatrix} 0 \\ 93.75 \\ 93.75 + 93.75 \\ 93.75 \\ 0 \end{Bmatrix}$$

$$[k]^G = \begin{bmatrix} 1.42125 & -0.28125 & -0.28125 & 0 & -0.86 \\ -0.28125 & 4.78 & -0.75 & 0 & 0 \\ -0.28125 & -0.75 & 9.5625 & -0.75 & -0.28125 \\ 0 & 0 & -0.75 & 4.78 & -0.28125 \\ -0.86 & 0 & -0.28125 & -0.28125 & 1.42125 \end{bmatrix}$$

$$\begin{bmatrix} 1.42125 & -0.28125 & -0.28125 & 0 & -0.86 \\ -0.28125 & 4.78 & -0.75 & 0 & 0 \\ -0.28125 & -0.75 & 9.5625 & -0.75 & -0.28125 \\ 0 & 0 & -0.75 & 4.78 & -0.28125 \\ -0.86 & 0 & -0.28125 & -0.28125 & 1.42125 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = 93.75 \begin{Bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 0 \end{Bmatrix}$$

Substitute for T_4 & T_5 as 180º and evaluate T_1, T_2, T_3

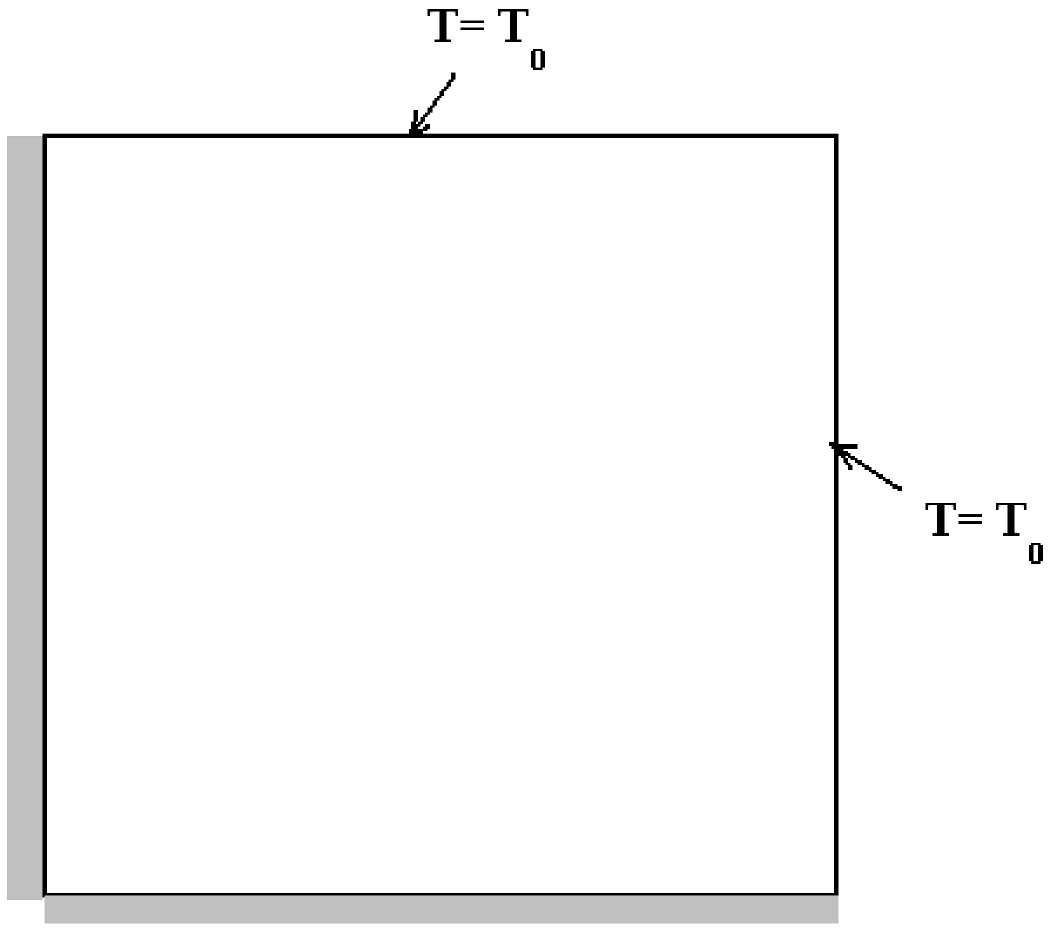
$$\begin{bmatrix} 1.42125 & -0.28125 & -0.28125 & 0 & -0.86 \\ -0.28125 & 4.78 & -0.75 & 0 & 0 \\ -0.28125 & -0.75 & 9.5625 & -0.75 & -0.28125 \\ 0 & 0 & 0.75 & 4.78 & 0.28125 \\ -0.86 & 0 & -0.28125 & -0.28125 & 1.42125 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} =$$

$$93.75 \begin{Bmatrix} 0 + 0.86 * 180 \\ 1 \\ 2 + (0.28125 + 0.75) * 180 \\ 1 \\ 0 \end{Bmatrix}$$

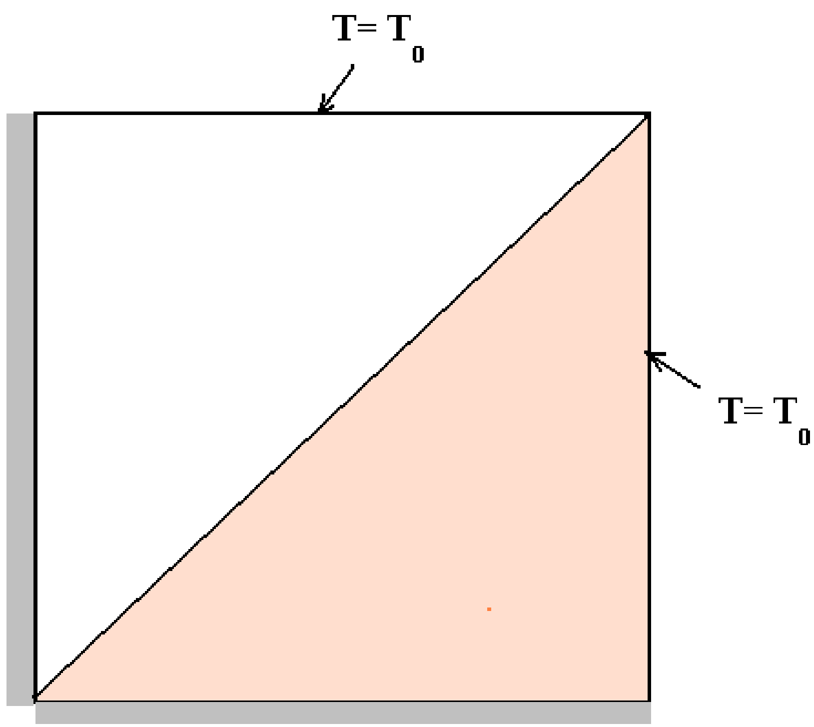
T_1 = 124.5ºC

T_2 = 34.0ºC

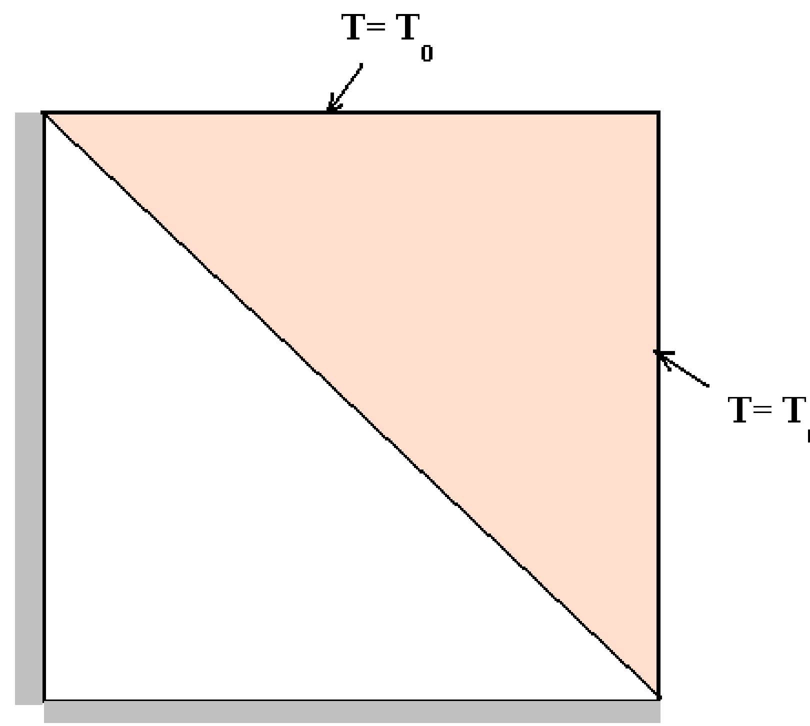
T_3 = 45.4ºC



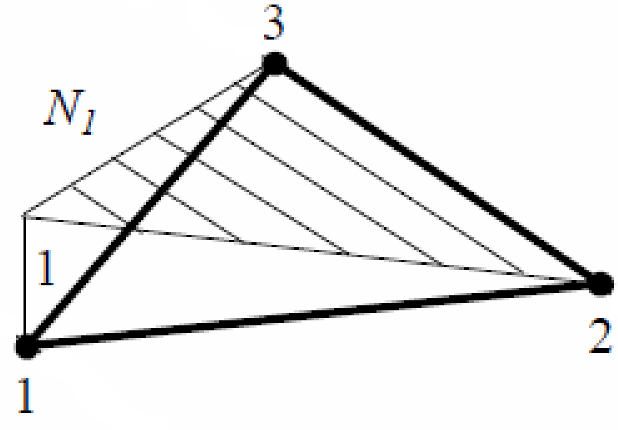
T= T0
T= T0



RIGHT



WRONG

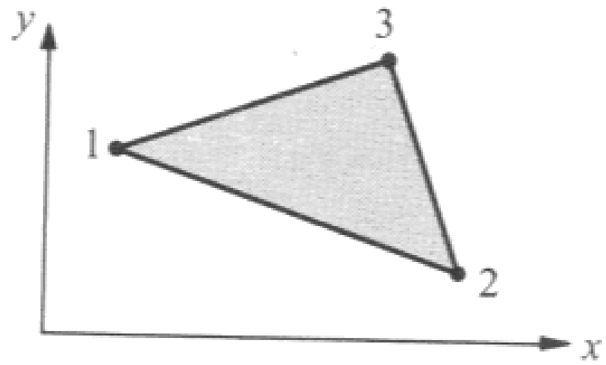


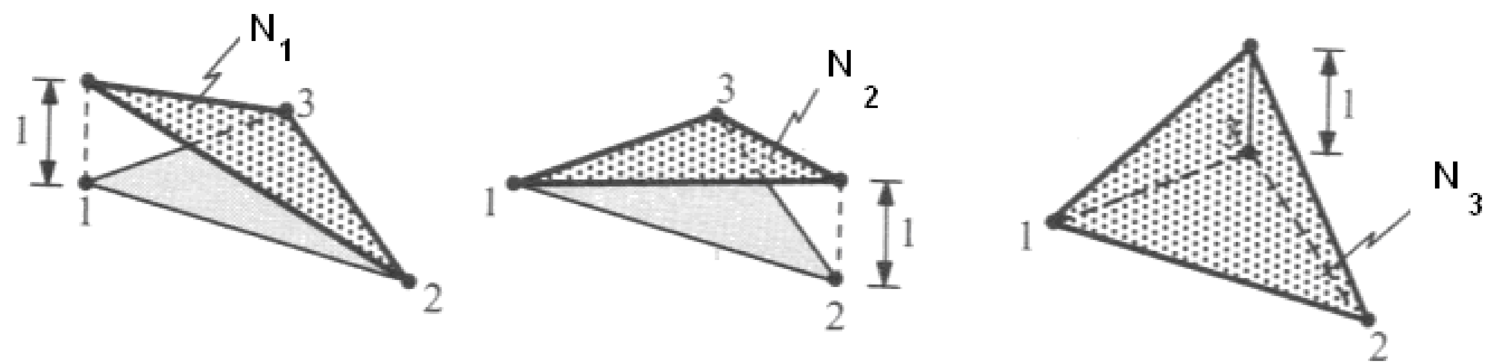
Shape Function N_1 for CST

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{\mathrm{yy}} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \varepsilon_{\mathrm{xy}} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_{xx} = \frac{1}{2A}(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3)$$

$$\varepsilon_{\mathrm{yy}} = \frac{1}{2A}(\gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3)$$





Variation of Shape functions for CST element

STIFFNESS MATRIX FOR BI LINEAR RECTANGULAR ELEMENT

$$N_1 = \left(1 - \frac{x}{2a}\right)\left(1 - \frac{y}{2b}\right) \qquad N_3 = \left(\frac{x}{2a}\right)\left(\frac{y}{2b}\right)$$

$$N_2 = \left(\frac{x}{2a}\right)\left(1 - \frac{y}{2b}\right) \qquad N_4 = \left(1 - \frac{x}{2a}\right)\left(\frac{y}{2b}\right)$$

$$k_{11} = \int_0^{2a}\int_0^{2b} \frac{dN_1}{dx} \cdot \frac{dN_1}{dx}\, dxdy$$

$$\frac{dN_1}{dx} = -\frac{1}{2a}\left(1 - \frac{y}{2b}\right) \qquad \frac{dN_3}{dx} = \frac{1}{2a}\left(\frac{y}{2b}\right)$$

$$\frac{dN_2}{dx} = \frac{1}{2a}\left(1 - \frac{y}{2b}\right) \qquad \frac{dN_4}{dx} = -\frac{1}{2a}\left(\frac{y}{2b}\right)$$

$$\therefore k_{11} = \int_0^{2a}\int_0^{2b} \frac{1}{4a^2}\left(1-\frac{y}{2b}\right)^2 dxdy$$

$$= \frac{b}{3a}$$

$$k_{12} = \int_0^{2a}\int_0^{2b} -\frac{1}{2a}\times\frac{1}{2a}\left(1-\frac{y}{2b}\right)^2 dxdy$$

$$= -\frac{b}{3a}$$

$$k_{13} = \int_0^{2a}\int_0^{2b} -\frac{1}{2a} \times \frac{1}{2a}\left(1 - \frac{y}{2b}\right) dxdy$$

$$= -\frac{b}{6a}$$

$$k_{14} = \int_0^{2a}\int_0^{2b} \frac{1}{4a^2}\left(\frac{y}{2b}\right)\left(1 - \frac{y}{2b}\right) dxdy$$

$$= \frac{b}{6a}$$

$$k_{22} = \int_0^{2a}\int_0^{2b} \frac{1}{4a^2}\left(1 - \frac{y}{2b}\right) dxdy$$

$$= \frac{b}{3a}$$

$$k_{23} = \int_0^{2a}\int_0^{2b} \frac{1}{4a^2}\left(1 - \frac{y}{2b}\right) dxdy$$

$$= \frac{b}{6a}$$

$$k_{24} = \int_0^{2a}\int_0^{2b} -\frac{1}{4a^2}\left(1-\frac{y}{2b}\right) dxdy$$

$$= -\frac{b}{6a}$$

$$k_{33} = \int_0^{2a}\int_0^{2b} \frac{1}{4a^2}\left(\frac{y^2}{4b^2}\right) dxdy$$

$$= \frac{b}{3a}$$

$$k_{34} = \int_0^{2a}\int_0^{2b} -\frac{1}{4a^2}\left(\frac{y^2}{4b^2}\right) dxdy$$

$$= -\frac{b}{3a}$$

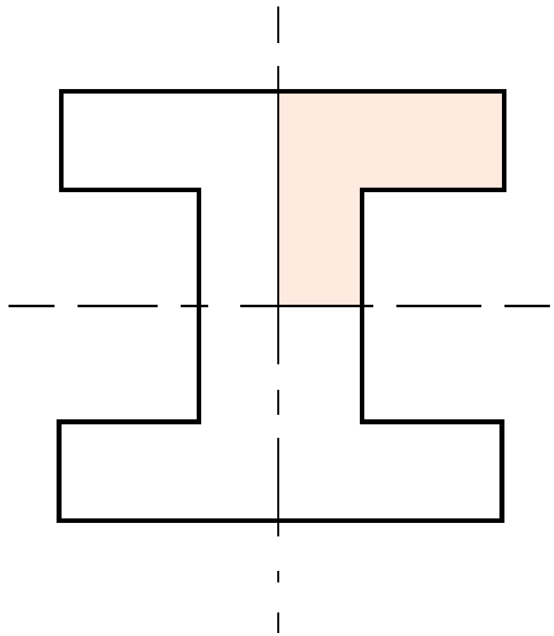
$$K_{44} = \int_0^{2a}\int_0^{2b} \frac{dN_4}{dx}\frac{dN_4}{dx} dxdy$$

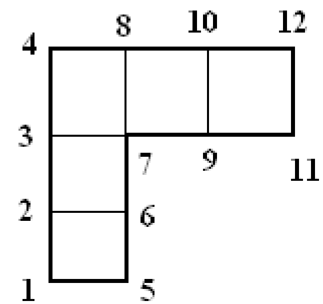
$$= \int_0^{2a}\int_0^{2b} \frac{1}{4a^2}\frac{y^2}{4b^2} dxdy = \frac{b}{3a}$$

$$[K]^{e} = \begin{bmatrix} \frac{b}{3a} & -\frac{b}{3a} & -\frac{b}{6a} & \frac{b}{6a} \\ -\frac{b}{3a} & \frac{b}{3a} & \frac{b}{6a} & -\frac{b}{6a} \\ -\frac{b}{6a} & \frac{b}{6a} & \frac{b}{3a} & -\frac{b}{3a} \\ \frac{b}{6a} & -\frac{b}{6a} & -\frac{b}{3a} & \frac{b}{3a} \end{bmatrix} + \begin{bmatrix} \frac{a}{3b} & \frac{a}{6b} & -\frac{a}{6b} & -\frac{a}{3b} \\ \frac{a}{6b} & \frac{a}{3b} & -\frac{a}{3b} & -\frac{a}{6b} \\ -\frac{a}{6b} & -\frac{a}{3b} & \frac{a}{3b} & \frac{a}{6b} \\ -\frac{a}{3b} & -\frac{a}{6b} & \frac{a}{6b} & \frac{a}{3b} \end{bmatrix}$$

$$=k\times\frac{b}{6a}\begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix}+k\times\frac{a}{6b}\begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix}$$

$$=\frac{k}{6ab}\begin{bmatrix} 2(a^2+b^2) & a^2-2b^2 & -(a^2+b^2) & (b^2-2a^2) \\ a^2-2b^2 & 2(a^2+b^2) & (b^2-2a^2) & -(a^2+b^2) \\ -(a^2+b^2) & (b^2-2a^2) & 2(a^2+b^2) & (-2b^2+a^2) \\ (b^2-2a^2) & -(a^2+b^2) & a^2-2b^2 & 2(a^2+b^2) \end{bmatrix}$$



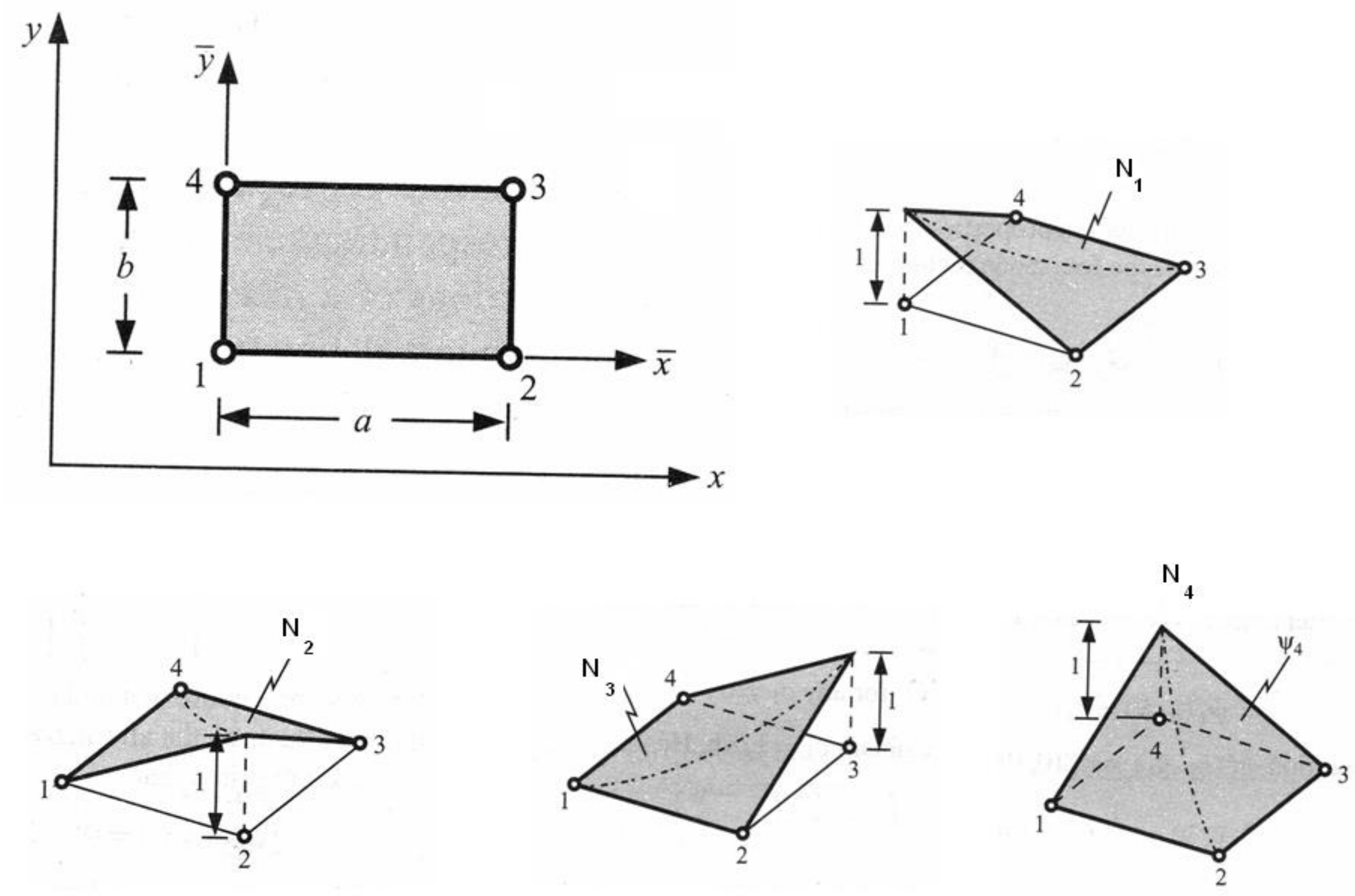

4
8
10
12
3
7
9
11
2
6
1
5

$$N_1 = \left(1 - \frac{x}{3}\right)\left(1 - \frac{y}{2}\right) \qquad N_3 = \frac{xy}{6}$$

$$N_2 = \frac{x}{3}\left(1 - \frac{y}{2}\right) \qquad N_4 = \frac{y}{2}\left(1 - \frac{x}{3}\right)$$

$$\frac{dN_1}{dx} = -\frac{1}{2a}\left(1 - \frac{y}{2b}\right) \qquad \frac{dN_3}{dx} = \frac{1}{2a}\left(\frac{y}{2b}\right)$$

$$\frac{dN_2}{dx} = \frac{1}{2a}\left(1 - \frac{y}{2b}\right) \qquad \frac{dN_4}{dx} = -\frac{1}{2a}\left(\frac{y}{2b}\right)$$



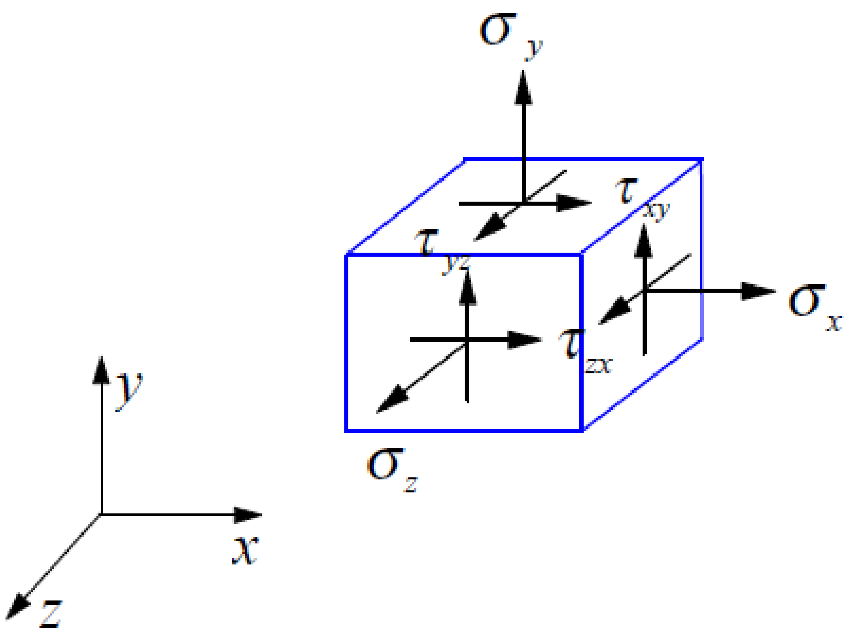
y
$\bar{y}$
4
3
b
1
2
$\bar{x}$
a
x
N_1
N_2
N_3
N_4
ψ_4

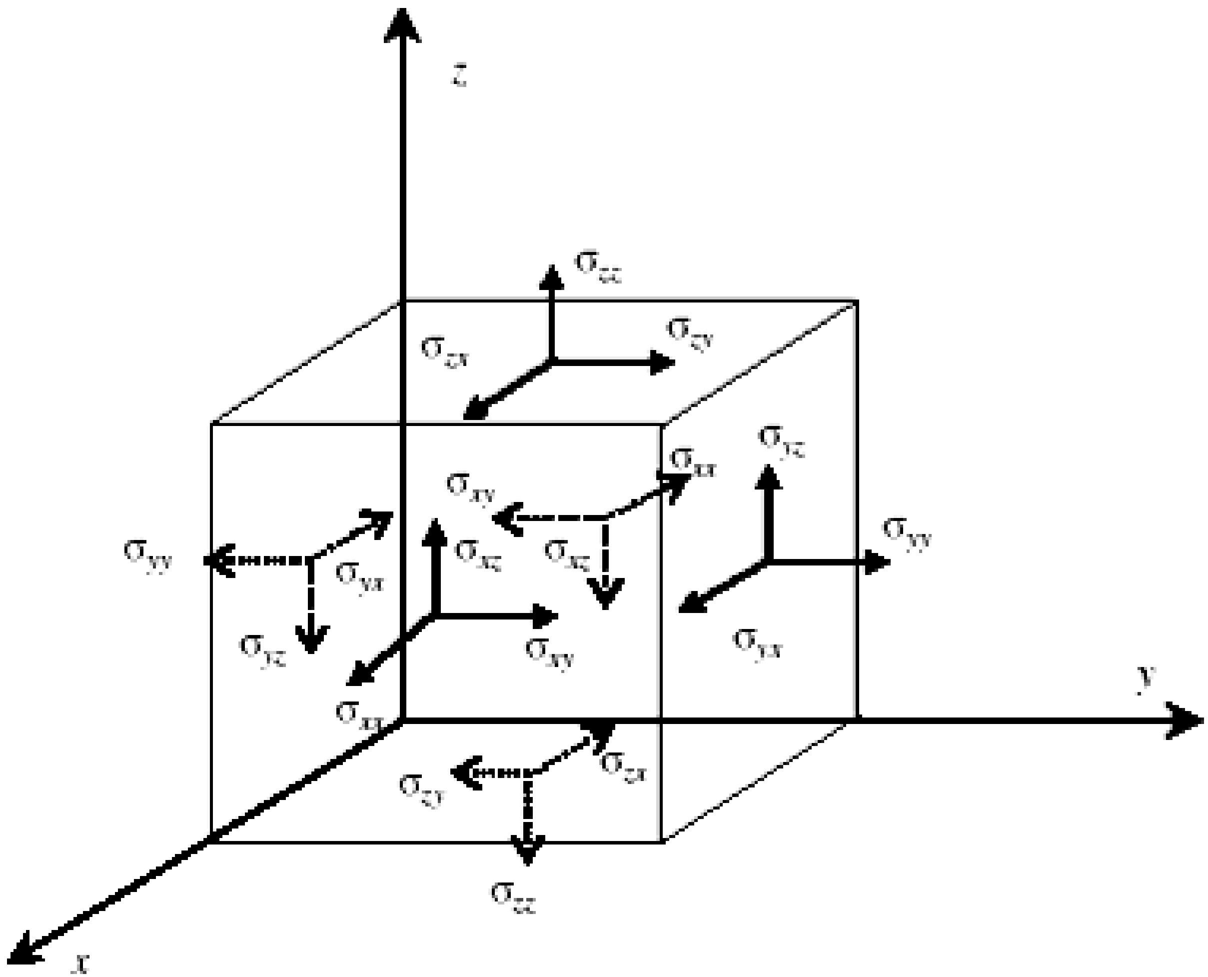
VECTOR VARIABLE PROBLEMS

$$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx} \qquad \text{for stresses,}$$

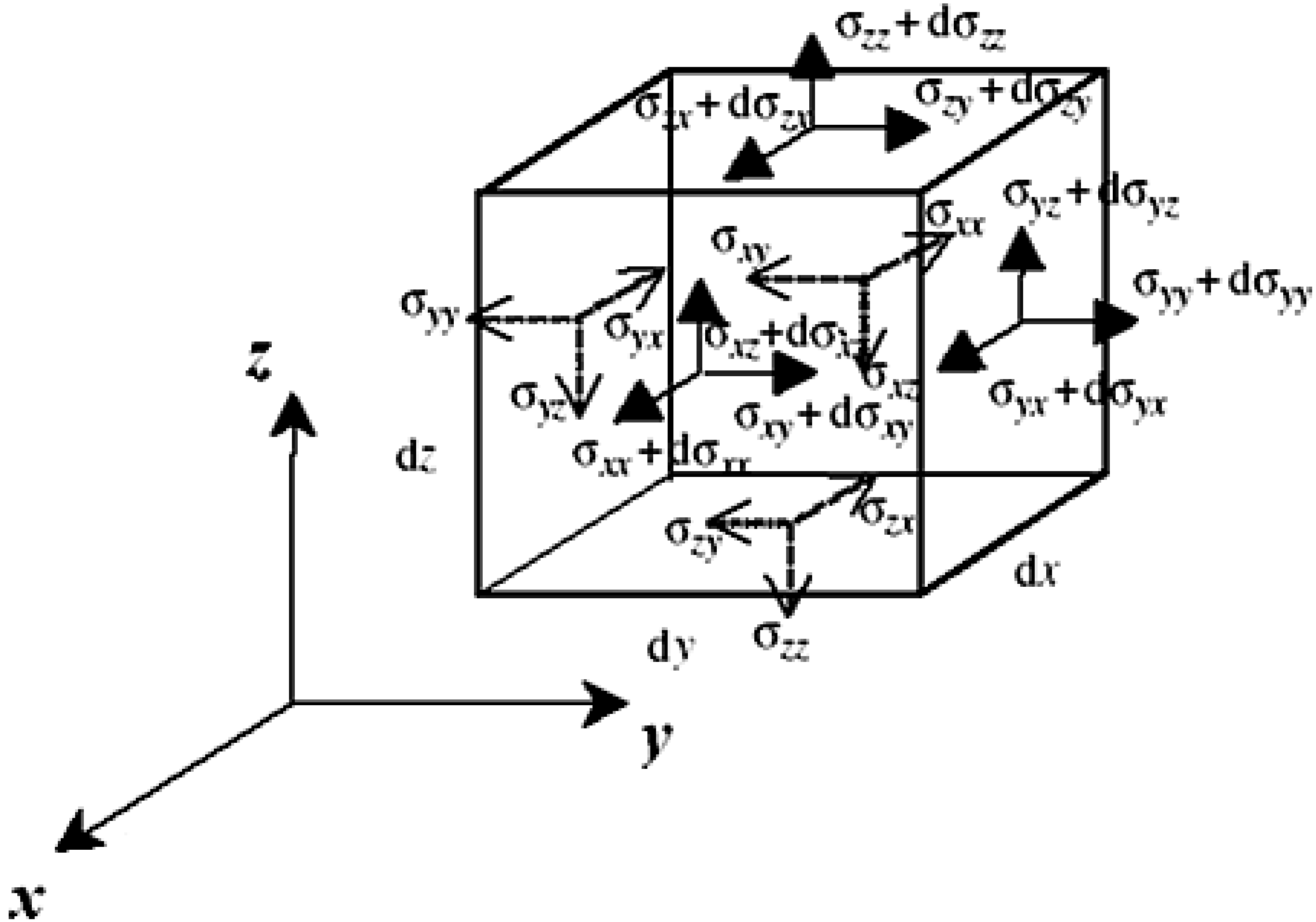
and

$$\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx} \qquad \text{for strains.}$$





Three dimensional stresses



Stresses on an elemental cuboid

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + B_x = 0$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + B_z = 0$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + B_y = 0$$

Force Equilibrium Equations

$\sum M_x = 0$,$\sum M_y = 0$ & $\sum M_z = 0$ yields

$\tau_{xy} = \tau_{yx}$; $\tau_{yz} = \tau_{zy}$; $\tau_{zx} = \tau_{xz}$ (2)

Strain – displacement relations:-

$$\in_{xx} = \frac{\partial u}{\partial x}$$

$$\in_{yy} = \frac{\partial v}{\partial y}$$

$$\in_{zz} = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Stress – Strain Relations:-

$$\in_{xx} = \frac{\sigma_{xx}}{E} - \frac{\mu}{E}(\sigma_{yy} + \sigma_{zz})$$

$$\in_{yy} = \frac{\sigma_{yy}}{E} - \frac{\mu}{E}(\sigma_{xx} + \sigma_{zz})$$

$$\in_{zz} = \frac{\sigma_{zz}}{E} - \frac{\mu}{E}(\sigma_{xx} + \sigma_{yy})$$

$$\gamma_{xy} = \tau_{xy} / G$$

$$\gamma_{yz} = \tau_{yz} / G$$

$$\gamma_{zx} = \tau_{zx} / G$$

Where E = Young's Modulus

G = Shear Modulus $= \dfrac{E}{2(1+\mu)}$

μ = Poisson's ratio

The equations (6) can be written in matrix form as

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\mu & -\mu & 0 & 0 & 0 \\ -\mu & 1 & -\mu & 0 & 0 & 0 \\ -\mu & -\mu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\mu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\mu) \end{pmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}$$

. $\{\in\} = [C] \quad \{\sigma\}$

$\therefore \quad \{\sigma\} = [C]-1 \{\in\}$

$= [D] \quad \{\in\}$

Here the matrix [D] is called the constitutive matrix given by

$$[D] = \frac{E}{1+\mu}\begin{pmatrix} \frac{1-\mu}{1-2\mu} & \frac{\mu}{1-2\mu} & \frac{\mu}{1-2\mu} & 0 & 0 & 0 \\ \frac{\mu}{1-2\mu} & \frac{1-\mu}{1-2\mu} & \frac{\mu}{1-2\mu} & 0 & 0 & 0 \\ \frac{\mu}{1-2\mu} & \frac{\mu}{1-2\mu} & \frac{1-\mu}{1-2\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{pmatrix} (1-\mu) & \mu & \mu & 0 & 0 & 0 \\ & (1-\mu) & \mu & 0 & 0 & 0 \\ & & (1-\mu) & 0 & 0 & 0 \\ & & & \frac{1-2\mu}{2} & 0 & 0 \\ \text{Symmetric} & & & & \frac{1-2\mu}{2} & 0 \\ & & & & & \frac{1-2\mu}{2} \end{pmatrix}$$

Strain and Displacement Relations

For small strains and small rotations, we have,

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

In matrix form,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \partial / \partial x & 0 \\ 0 & \partial / \partial y \\ \partial / \partial y & \partial / \partial x \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad \text{or} \quad \varepsilon = \Lambda \ \mathbf{u}$$

From this relation, we know that the strains (and thus stresses) are one order lower than the displacements, if the displacements are represented by polynomials.

Displacements (u, v) in a plane element are interpolated from nodal displacements (u_i, v_i) using shape functions N_i as follows,

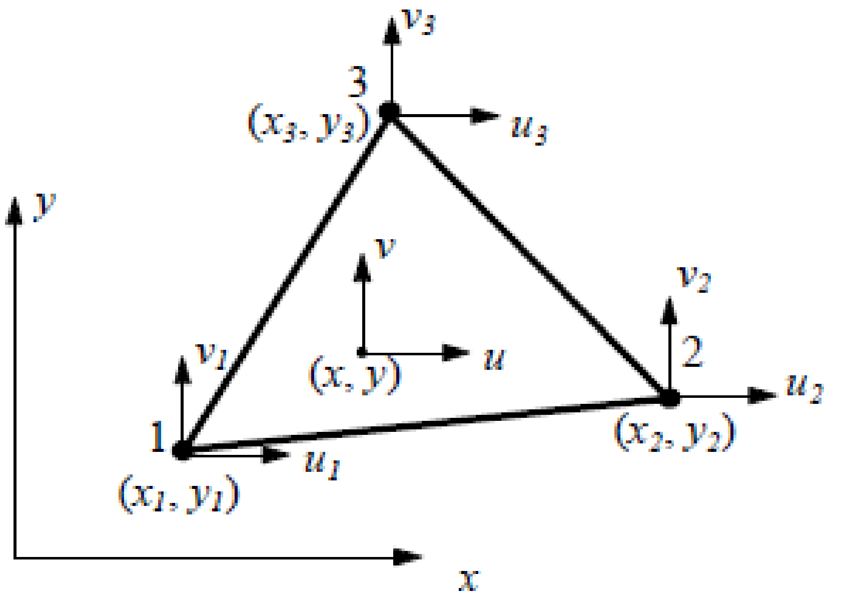
$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots \\ 0 & N_1 & 0 & N_2 & \cdots \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{Bmatrix} \quad \text{or} \quad \mathbf{u} = \mathbf{N}\mathbf{d} \qquad (11)$$

where $\mathbf{N}$ is the *shape function matrix*, $\mathbf{u}$ the displacement vector and $\mathbf{d}$ the *nodal* displacement vector. Here we have assumed that u depends on the nodal values of u only, and v on nodal values of v only.

From strain-displacement relation (Eq.(8)), the strain vector is,

$$\varepsilon = \Lambda \mathbf{u} = \Lambda \mathbf{N}\mathbf{d}, \qquad \text{or} \qquad \varepsilon = \mathbf{B}\mathbf{d}$$

where $\mathbf{B} = \Lambda \mathbf{N}$ is the *strain-displacement matrix*.



Linear Triangular Element

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

where the shape functions (linear functions in x and y) are

$$N_1 = \frac{1}{2A}\{(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y\}$$

$$N_2 = \frac{1}{2A}\{(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y\}$$

$$N_3 = \frac{1}{2A}\{(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y\}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = Bd = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Stress-Strain Relations

For elastic and isotropic materials, we have,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$\varepsilon = \mathbf{E}^{-1}\sigma$$

where E the Young's modulus, ν the Poisson's ratio and G the shear modulus.

Note that, $$G = \frac{E}{2(1+\nu)}$$

$\{\sigma\} = [D] \quad \{\in\} = DBd$

STRAIN DISPLACEMENT RELATIONS

$$\{\in\} = \Lambda u = B\ d$$

Where B = = ΛN

STRESS STRAIN RELATIONS

$$\{\sigma\} = [D]\ \{\in\} = DBd$$

2-D APPROXIMATIONS OF 3 – D PROBLEMS

There exists several problems in solid mechanics that can be formulated as three Dimensional problems and the finite element technique can be used to solve them.

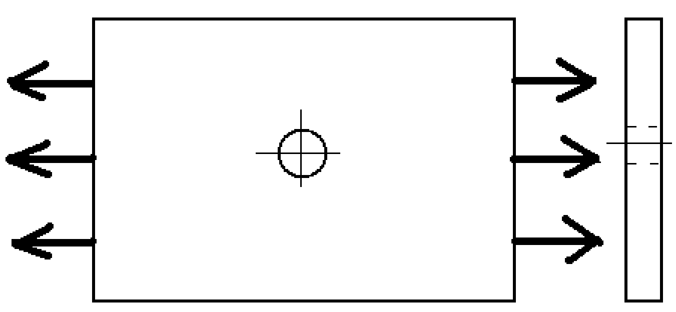
- **However it may turn out to be costly and time consuming to perform Finite Element Analysis of 3 D problems.**

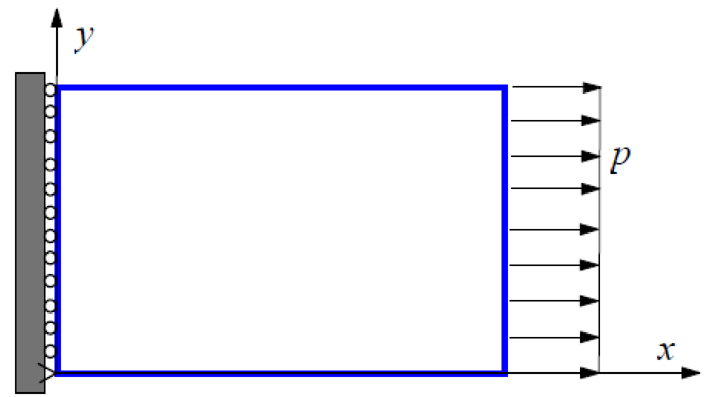
- **In several practical situations the geometry and loading may be such that the problem can be reduced from 3 D to 2 D or from 2D to 1D.**

- **The two dimensional idealizations in stress analysis include**

i. **PLANE STRESS problems**
ii. **PLANE STRAIN problems**
iii. **AXISYMMETRIC problems**

PLANE STRESS: - A 3D problem can be reduced to a plane stress condition if it is characterized by very small dimensions in one of the normal directions.





A thin plate with a cut out subjected to in-plane loading.

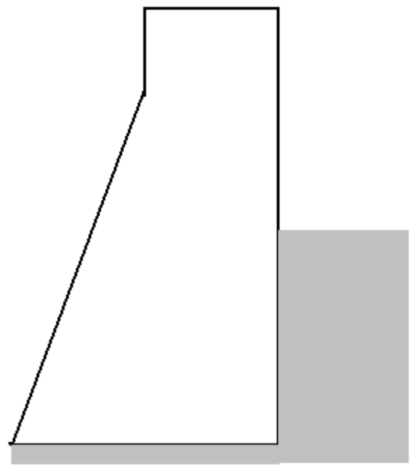
Thin plate subjected to in-plane loading

In these cases the stress components σ_z, τ_{xz}, & τ_{yz} are zero and it is assumed that no stress component varies across the thickness. The state of stress is then specified by σ_x, σ_y and τ_{xy} only, (functions of x & y) and is called plane stress. The stress strain relations are given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \tau_{xy} \end{Bmatrix}$$

PLANE STRAIN:- There exist problems involving very long bodies i.e. a body whose geometry and loading do not vary significantly in the longitudinal direction. Such problems are referred to as plane strain problems.

Some typical examples include a long cylinder such as a tunnel, culvert or buried pipe, a laterally loaded retaining wall, a long earth dam, and a loaded semi-infinite half space such as a strip footing on a soil mass.



A long dam

In all these problems, the dependant variable can be assumed to be functions of only x & y co-ordinates provided that we consider a cross-section some distance away from the two ends.

If we further assume that ‘w’ the displacement component in the ‘z’ direction is zero at every cross-section, then the non-zero strain components will be

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad ; \varepsilon_y = \frac{\partial v}{\partial y} \quad ; \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

and the strain components

ε_z, γ_{xz}, γ_{yz} will vanish. The dependant stress variables are σ_x, σ_y & τ_{xy} and the constitutive relation for an elastic isotropic material is given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} (1-\mu) & \mu & 0 \\ \mu & (1-\mu) & 0 \\ 0 & 0 & \frac{(1-2\mu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \tau_{xy} \end{Bmatrix}$$

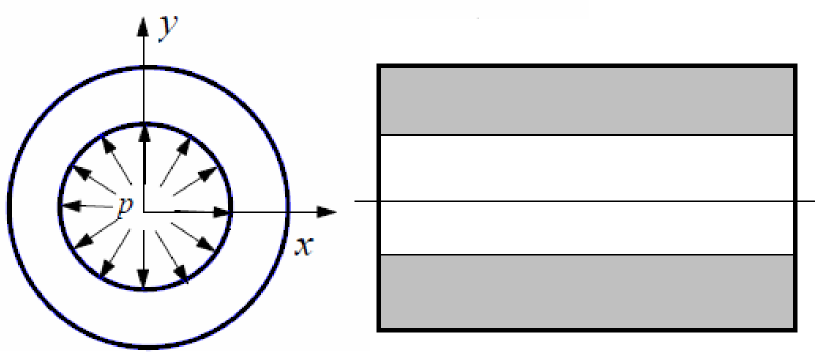
It is important to note here that only $\varepsilon_z = 0$ but $\sigma_z \neq 0$.

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\mu}{E}\sigma_x - \frac{\mu}{E}\sigma_y = 0$$

$$\therefore \sigma_z = -\mu \; (\sigma_x + \sigma_y)$$

AXISYMMETRIC PROBLEMS:- Many engineering problems involve solids of revolution (axisymmetric solids) subject to axially symmetric loading.

Examples are a circular cylinder loaded by uniform internal or external pressure or other axially symmetric loading as shown in



and a semi – infinite half space loaded by a circular area. eg., a circular footing on a soil mass.

Because of symmetry the stress components are independent of the angular co-ordinate **'θ'** and hence all the derivatives with respect to 'θ ' vanish and the components υ, υ_{θ},$\upsilon_{\theta z}$, $\tau_{x\theta}$,$\tau_{\theta y}$ are zero. The strain displacement relation are given by

$$\varepsilon_r = \frac{\partial u}{\partial x} \ ; \ \varepsilon_\theta = \frac{u}{r} \ ; \ \varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

The constitutive relations is

Stresses:

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \dfrac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{Bmatrix}$$

Now the strain energy stored in an element is given by

$U = \frac{1}{2} \int_v \{ \varepsilon \}^T \{ \sigma \} dv$

$\quad = \frac{1}{2} \int_v \{ \varepsilon \}^T [D] \{ \varepsilon \} dv$

$\quad = \frac{1}{2} \int_v [B]^T \{d\} [D] [B] \{d\} dv$

The work done by nodal forces is given by

$W = \frac{1}{2} \int_v \{F\} \{d\} dv$

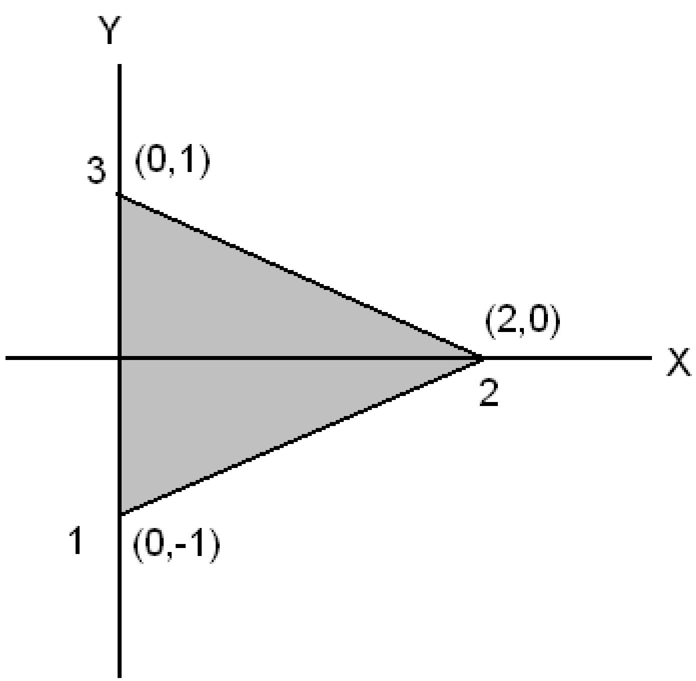
Equating for a conservative system we get

$\int_v ([B]^T [D] [B]) dv \{d\} = \{F\}$

i.e. $[K] \{d\} = \{F\}$

where $[K] = \int_v [B]^T [D] [B] \, dv$

Problem 2:- Assuming plane stress conditions evaluate the stiffness matrix for the element shown in Fig. Assume E= 2 x 105 N/cm^2 and μ= 0.3. u1 = 0.000, v_1 = 0.0025, u_2 = 0.0012, v_2 = 0.000, u_3 = 0.0000 & v_3 = 0.0025.



$\beta_1 = y_2 – y_3 \quad = 0 – 1 = -1$

$\beta_2 = y_3 – y_1 \quad = 1 + 1 = 2$

$\beta_3 = y_1 – y2 \quad = -1– 0 = -1$

$\gamma_1 = -(x_2 – x_3) = 0 – 2 = -2$

$\gamma_2 = -(x_3 – x_1) = 0 – 0 = 0$

$\gamma_3 = -(x_1 – x_2) = 2 – 0 = 2$

. $A = \frac{1}{2} \times b \times h = \frac{1}{2} \times 2 \times 2 = 2$

$$\{\in\} = \frac{1}{2A} \begin{pmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{pmatrix} \begin{Bmatrix} u1 \\ v1 \\ u2 \\ v2 \\ u3 \\ v3 \end{Bmatrix}$$

$$= \{B] \{d\}$$

$$[B] = \frac{1}{2(2)} \begin{pmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{pmatrix}$$

$$[D] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

$$= \frac{2 \times 10^5}{1-(0.3)^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

Now we know that the stiffness matrix [K] is given by $\int_{vol} [B]^T [D] [B] dv$

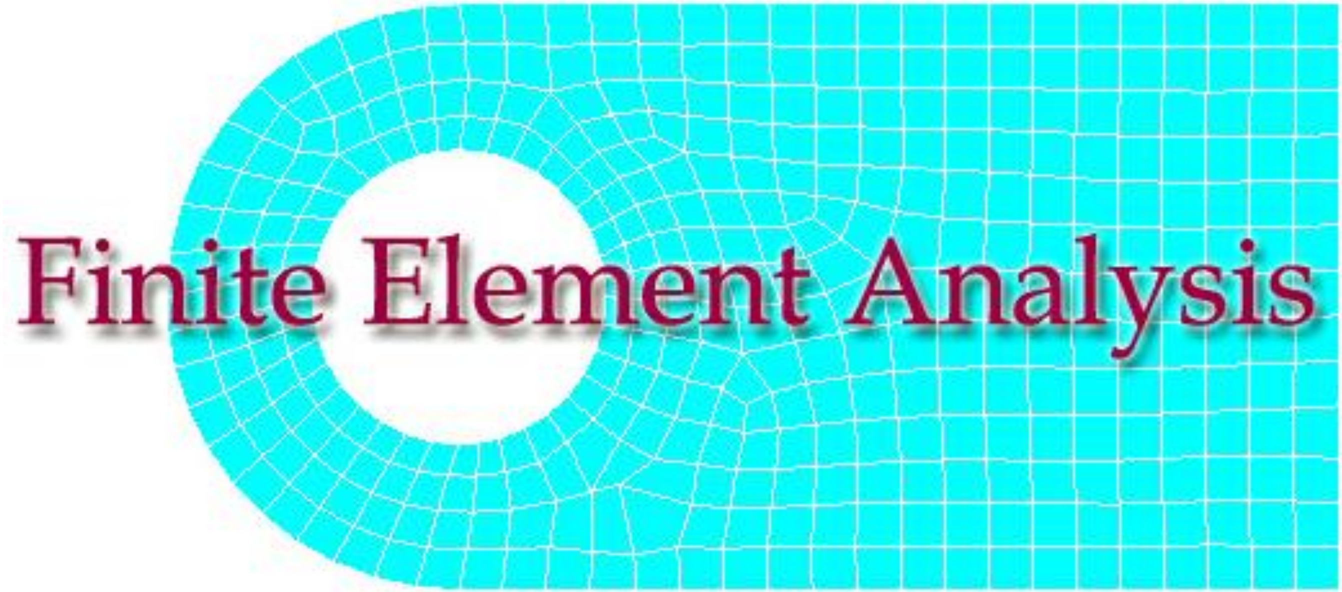
Assuming unit thickness ie t = 1 we get

$[K] = A\,[B]^T [D] \{B]$

$$= \frac{(2)(2 \times 10^5)}{4\,(0.91)} \underset{6 \times 3}{\begin{pmatrix} -1 & 0 & -2 \\ 0 & -2 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{pmatrix}} \underset{3 \times 3}{\begin{pmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 1 & 0.35 \end{pmatrix}} \; \frac{1}{4} \underset{3 \times 6}{\begin{pmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{pmatrix}}$$

$$= 1.099 \times 10^2 \begin{pmatrix} 600 & 325 & -500 & -350 & -100 & 25 \\ 325 & 1087.5 & -300 & -175 & -25 & -912.5 \\ -500 & -300 & 1000 & 0 & -500 & 300 \\ -350 & -175 & 0 & 350 & 350 & -175 \\ -100 & -25 & -500 & 350 & 600 & -325 \\ 25 & -912.5 & 300 & -175 & -325 & 1087.5 \end{pmatrix}_{6 \times 6}$$

Note: In order to evaluate the element stress we can use the equation $\{\sigma\} = [D]\,[B]\,\{d\}$



TWO DIMENSIONAL ELEMENTS- VECTOR VARIABLES

LECTURE 10

Types of 2D Problems

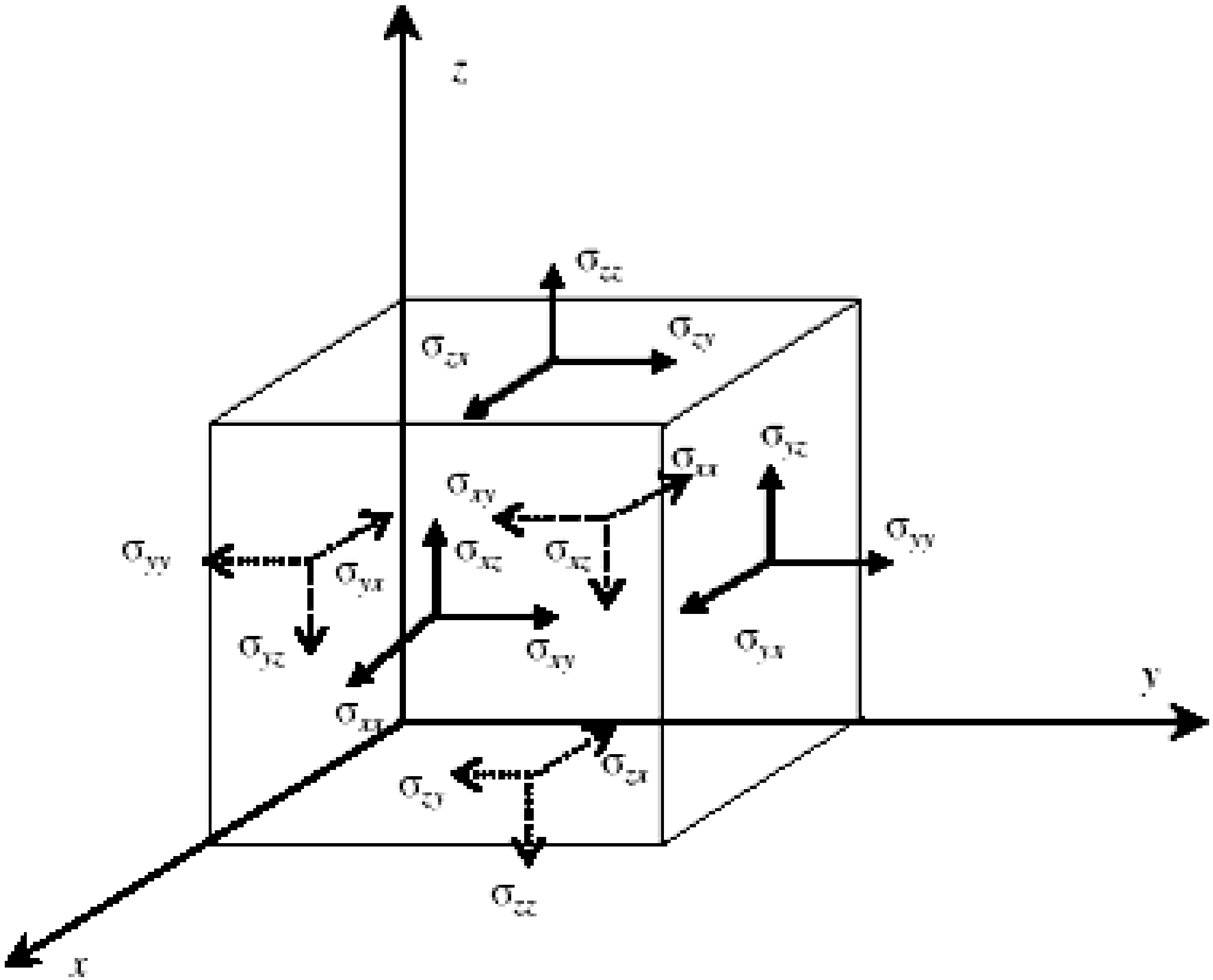
- **VECTOR VARIABLE PROBLEMS**

  e.g. Structural problems

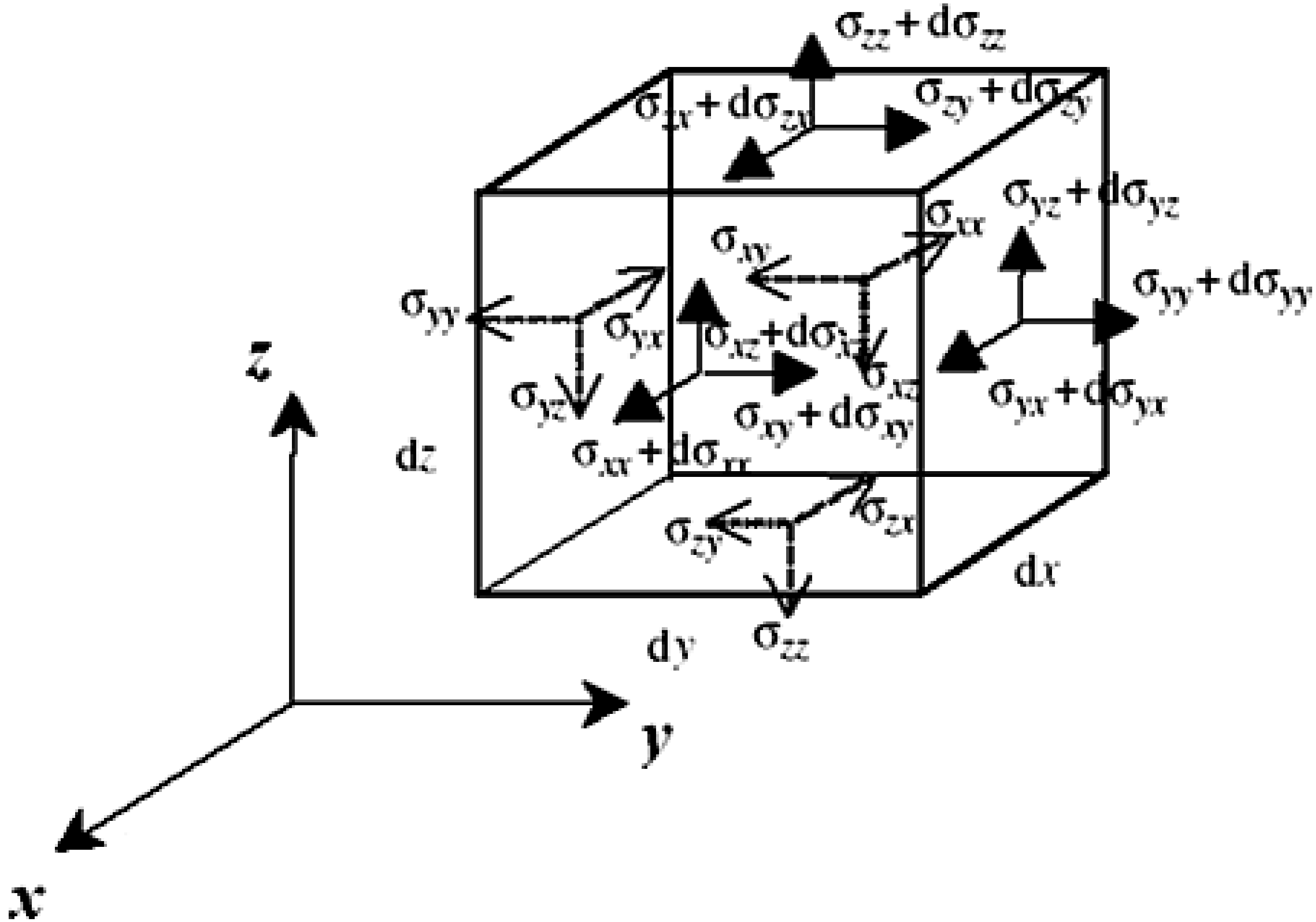
- **SCALAR VARIABLE PROBLEMS**

  e.g. Torsion of non-circular shafts,

  Heat transfer through fins



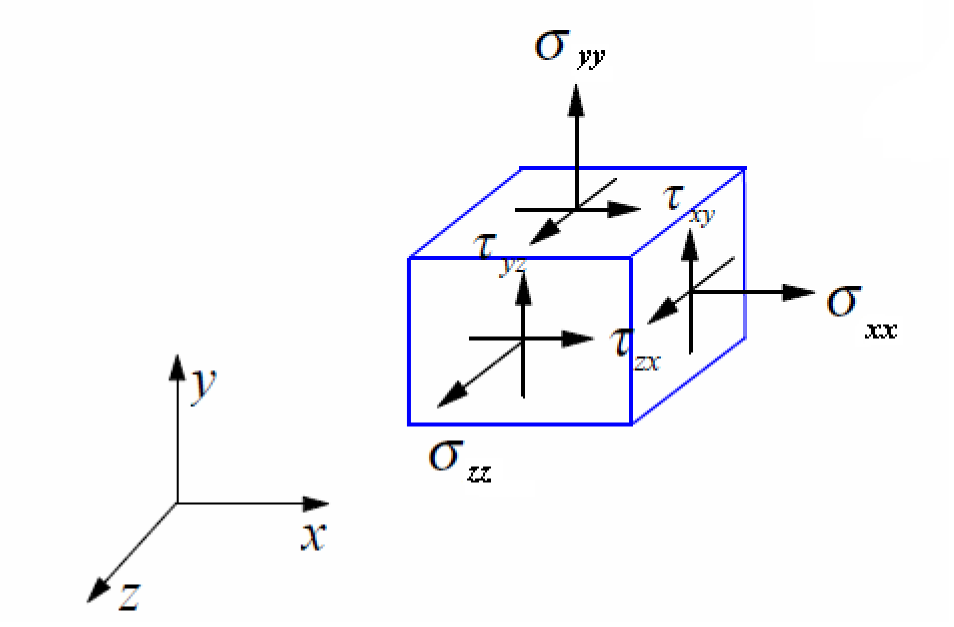
Three dimensional stresses



Stresses on an elemental cuboid

Stresses in 3D

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yx} \\ \tau_{yz} \\ \tau_{zy} \\ \tau_{xz} \\ \tau_{zx} \end{Bmatrix}$$



$$\left.\begin{aligned}
\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+B_x &= 0\\
\frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+B_y &= 0\\
\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+B_z &= 0
\end{aligned}\right\}$$

Force Equilibrium Equations

$\sum M_x = 0$, $\sum M_y = 0$ & $\sum M_z = 0$ yields

$$\tau_{xy} = \tau_{yx} \;;\; \tau_{yz} = \tau_{zy} \;;\; \tau_{zx} = \tau_{xz} \qquad (2)$$

Stresses in 3D

$$\{\sigma\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}$$

Strains in 3D

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

Strain – displacement relations:-

$$\epsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Stress – Strain Relations:-

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\mu}{E}(\sigma_{yy} + \sigma_{zz})$$

$$\epsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\mu}{E}(\sigma_{xx} + \sigma_{zz})$$

$$\epsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\mu}{E}(\sigma_{xx} + \sigma_{yy})$$

$$\gamma_{xy} = \tau_{xy} / G$$

$$\gamma_{yz} = \tau_{yz} / G$$

$$\gamma_{zx} = \tau_{zx} / G$$

Where

E = Young's Modulus

G = Shear Modulus $= \dfrac{E}{2(1+\mu)}$

μ = Poisson's ratio

The equations (6) can be written in matrix form as

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\mu & -\mu & 0 & 0 & 0 \\ -\mu & 1 & -\mu & 0 & 0 & 0 \\ -\mu & -\mu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\mu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\mu) \end{pmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}$$

. $\{\in\} = [C] \ \{\sigma\}$

$\therefore \ \{\sigma\} = [C]^{-1} \{\in\}$

$= [D] \ \{\in\}$

Here the matrix [D] is called the constitutive matrix given by

$$[D] = \frac{E}{1+\mu}\begin{pmatrix} \frac{1-\mu}{1-2\mu} & \frac{\mu}{1-2\mu} & \frac{\mu}{1-2\mu} & 0 & 0 & 0 \\ \frac{\mu}{1-2\mu} & \frac{1-\mu}{1-2\mu} & \frac{\mu}{1-2\mu} & 0 & 0 & 0 \\ \frac{\mu}{1-2\mu} & \frac{\mu}{1-2\mu} & \frac{1-\mu}{1-2\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{pmatrix} (1-\mu) & \mu & \mu & 0 & 0 & 0 \\ & (1-\mu) & \mu & 0 & 0 & 0 \\ & & (1-\mu) & 0 & 0 & 0 \\ & & & \frac{1-2\mu}{2} & 0 & 0 \\ \text{Symmetric} & & & & \frac{1-2\mu}{2} & 0 \\ & & & & & \frac{1-2\mu}{2} \end{pmatrix}$$

STRAIN DISPLACEMENT RELATIONS IN 2D

For small strains and small rotations, we have,

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

In matrix form,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \partial / \partial x & 0 \\ 0 & \partial / \partial y \\ \partial / \partial y & \partial / \partial x \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad \text{or} \quad \varepsilon = \Lambda \ \mathbf{u}$$

From this relation, we know that the strains (and thus stresses) are one order lower than the displacements, if the displacements are represented by polynomials.

Displacements (u, v) in a plane element are interpolated from nodal displacements (u_i, v_i) using shape functions N_i as follows,

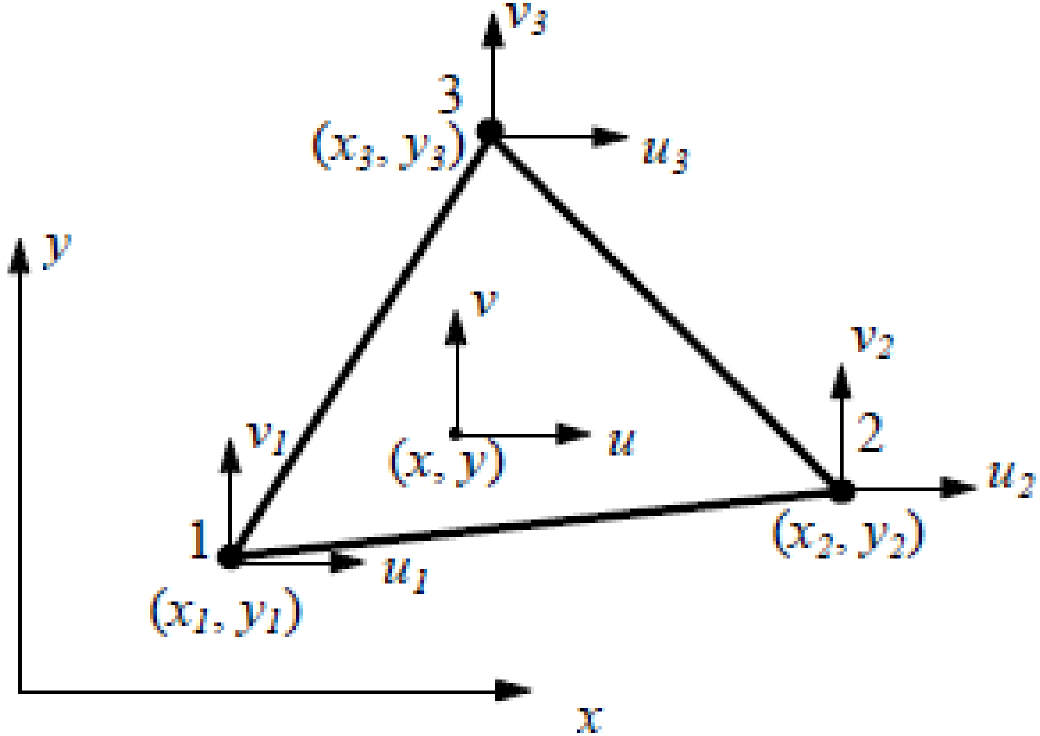
$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots \\ 0 & N_1 & 0 & N_2 & \cdots \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{Bmatrix} \quad \text{or} \quad \mathbf{u} = \mathbf{N}\mathbf{d} \quad (11)$$

where $\mathbf{N}$ is the *shape function matrix*, $\mathbf{u}$ the displacement vector and $\mathbf{d}$ the *nodal* displacement vector. Here we have assumed that u depends on the nodal values of u only, and v on nodal values of v only.

From strain-displacement relation (Eq.(8)), the strain vector is,

$$\varepsilon = \Lambda \mathbf{u} = \Lambda \mathbf{N}\mathbf{d}, \qquad \text{or} \qquad \varepsilon = \mathbf{B}\mathbf{d}$$

where $\mathbf{B} = \Lambda \mathbf{N}$ is the *strain-displacement matrix*.



Linear Triangular Element

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

where the shape functions (linear functions in x and y) are

$$N_1 = \frac{1}{2A}\{(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y\}$$

$$N_2 = \frac{1}{2A}\{(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y\}$$

$$N_3 = \frac{1}{2A}\{(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y\}$$

$$\boxed{N_i\,(x,y) \;=\; \frac{1}{2A_e}\,(\alpha_i \;+\; \beta_i\,x \;+\; \gamma_i\;y)}$$

$$B = \Lambda N = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = Bd = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Stress-Strain Relations

For elastic and isotropic materials, we have,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$\varepsilon = \mathbf{E}^{-1}\sigma$$

where E the Young's modulus, ν the Poisson's ratio and G the shear modulus.

Note that, $$G = \frac{E}{2(1+\nu)}$$

$\{\sigma\} = [D] \quad \{\in\} = DBd$

STRAIN DISPLACEMENT RELATIONS

$$\{\in\} = \Lambda u = B\ d$$

Where $B = = \Lambda N$

STRESS STRAIN RELATIONS

$$\{\sigma\} = [D]\ \{\in\} = DBd$$

Now the strain energy stored in an element is given by

$$U = \frac{1}{2}\int_{v} \varepsilon^{T}\sigma \; dv = \frac{1}{2}\int_{v} \varepsilon^{T} D\varepsilon dv$$

$$= \frac{1}{2}\int_{v} B^{T} d^{T} DBddv$$

$$\boxed{\varepsilon = Bd \;\&\; \sigma = DBd}$$

The work done by nodal forces is given by

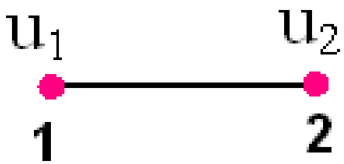
$$W = \frac{1}{2}\int_v F d^T \; dv$$

Equating strain energy to work done, for a conservative system we get

$$\frac{1}{2}\int_v B^T d^T D B d dv = \frac{1}{2}\int_v F d^T dv$$

$$ie.[K]\{d\} = \{F\}$$

$$where [K] = \int_v B^T D B dv$$



2 NODED LINEAR ELEMENT

$$N_1(x) = 1 - x/\ell$$

$$N_2(x) = x/\ell$$

$$\frac{dN_1}{dx} = \frac{-1}{l}$$

$$\frac{dN_2}{dx} = \frac{1}{l}$$

$$B = \left\langle \frac{-1}{l} \quad \frac{1}{l} \right\rangle$$

$$B = \left\langle \frac{-1}{l} \quad \frac{1}{l} \right\rangle \qquad B^T = \begin{Bmatrix} \dfrac{-1}{l} \\ \dfrac{1}{l} \end{Bmatrix}$$

$$K = \int_v B^T DBdv = \int_v \begin{Bmatrix} \dfrac{-1}{l} \\ \dfrac{1}{l} \end{Bmatrix} E < \frac{-1}{l} \frac{1}{l} > Adx =$$

$$\frac{EA}{l^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_0^l dx = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

2-D APPROXIMATIONS OF 3 – D PROBLEMS

There exists several problems in solid mechanics that can be formulated as three Dimensional problems and the finite element technique can be used to solve them.

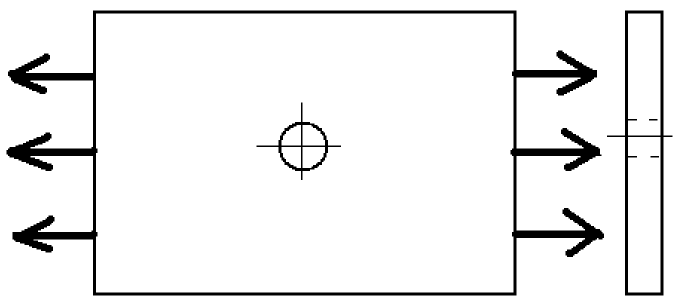
- **However it may turn out to be costly and time consuming to perform Finite Element Analysis of 3 D problems.**

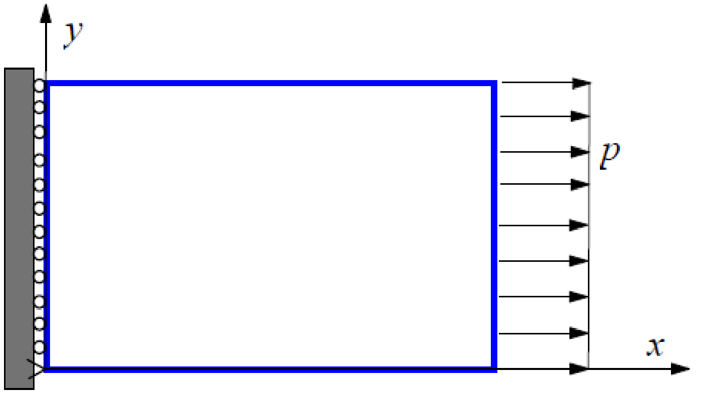
- **In several practical situations the geometry and loading may be such that the problem can be reduced from 3 D to 2 D or from 2D to 1D.**

- **The two dimensional idealizations in stress analysis include**

i. **PLANE STRESS problems**
ii. **PLANE STRAIN problems**
iii. **AXISYMMETRIC problems**

PLANE STRESS: - A 3D problem can be reduced to a plane stress condition if it is characterized by very small dimensions in one of the normal directions.





Eg.

A thin plate with a cut out subjected to in-plane loading.

Thin plate subjected to in-plane loading

In these cases the stress components σ_z, τ_{xz}, & τ_{yz} are zero and it is assumed that no stress component varies across the thickness. The state of stress is then specified by σ_x, σ_y and τ_{xy} only, (functions of x & y) and is called plane stress. The stress strain relations are given by

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\mu}{E}\sigma_{yy}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\mu}{E}\sigma_{xx}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{\tau_{xy}}{E}\,\frac{2(1+\mu)}{}$$

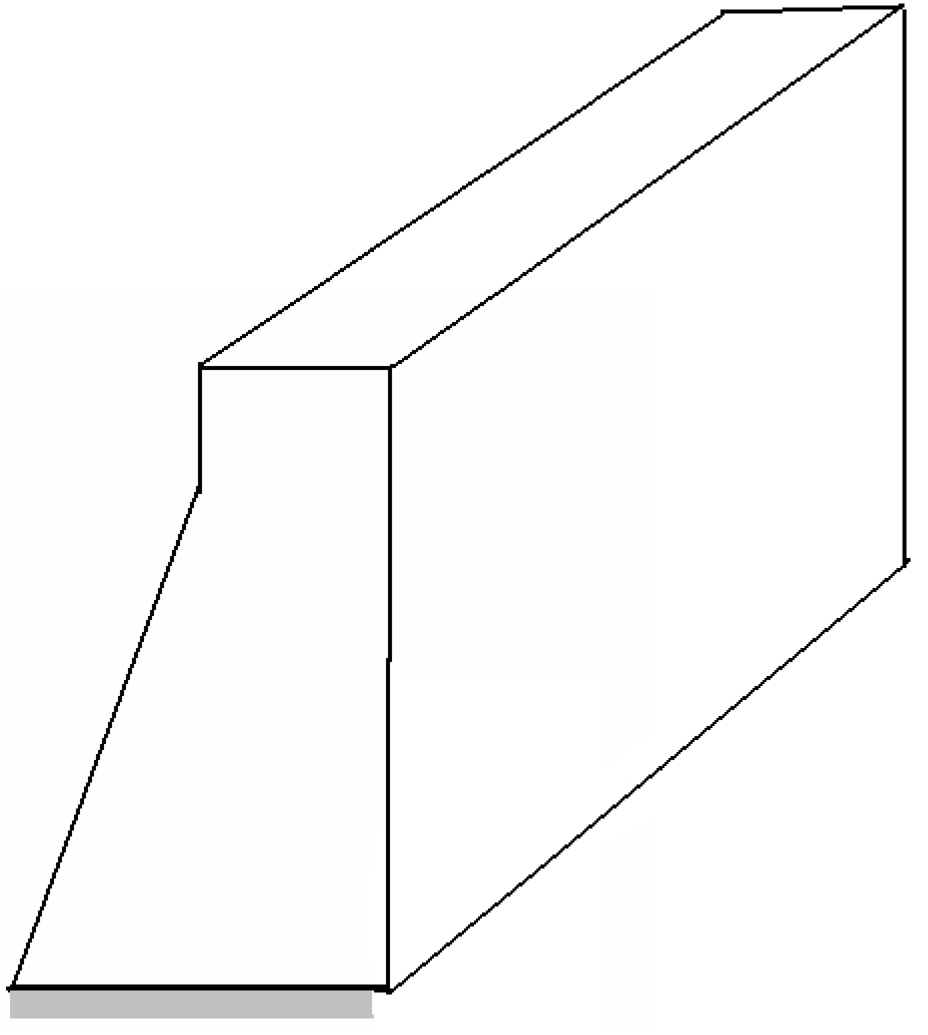
$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\mu}{E} & 0 \\ -\frac{\mu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{2(1+\mu)}{E} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\mu}{E} & 0 \\ -\frac{\mu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{2(1+\mu)}{E} \end{bmatrix}^{-1} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

PLANE STRAIN:- There exist problems involving very long bodies i.e. a body whose geometry and loading do not vary significantly in the longitudinal direction. Such problems are referred to as plane strain problems.

Some typical examples include a long cylinder such as a tunnel, culvert or buried pipe, a laterally loaded retaining wall, a long earth dam, and a loaded semi-infinite half space such as a strip footing on a soil mass.



A long dam

In all these problems, the dependant variable can be assumed to be functions of only x & y co-ordinates provided that we consider a cross-section some distance away from the two ends.

If we further assume that ‘w’ the displacement component in the ‘z’ direction is zero at every cross-section, then the non-zero strain components will be

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad ; \varepsilon_y = \frac{\partial v}{\partial y} \quad ; \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

and the strain components

ε_z, γ_{xz}, γ_{yz} will vanish. The dependant stress variables are σ_x, σ_y & τ_{xy} and the constitutive relation for an elastic isotropic material is given by

It is important to note here that only $\varepsilon_z = 0$ but $\sigma_z \neq 0$.

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\mu}{E}\sigma_x - \frac{\mu}{E}\sigma_y = 0$$

$$\therefore \sigma_z = \mu\ (\sigma_x + \sigma_y)$$

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\mu}{E}\sigma_{yy}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\mu}{E}\sigma_{xx}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{E}\frac{2(1+\mu)}{}$$

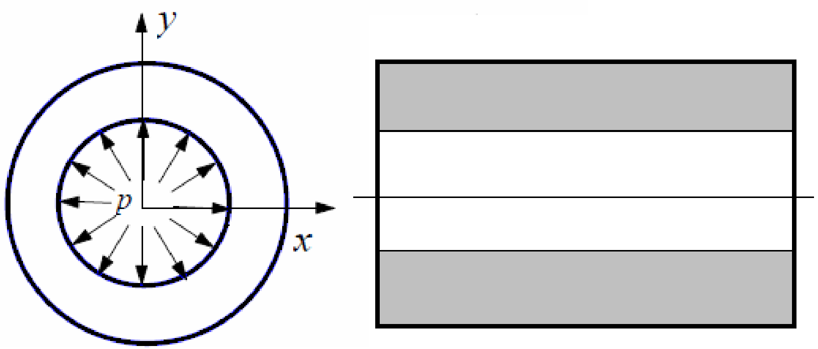
Substituting $\sigma_z = \mu \ (\sigma_x + \sigma_y)$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{pmatrix} (1-\mu) & \mu & 0 \\ \mu & (1-\mu) & 0 \\ 0 & 0 & \frac{(1-2\mu)}{2} \end{pmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

This is the constitutive matrix for plane strain element

AXISYMMETRIC PROBLEMS:- Many engineering problems involve solids of revolution (axisymmetric solids) subject to axially symmetric loading.

Examples are a circular cylinder loaded by uniform internal or external pressure or other axially symmetric loading as shown in



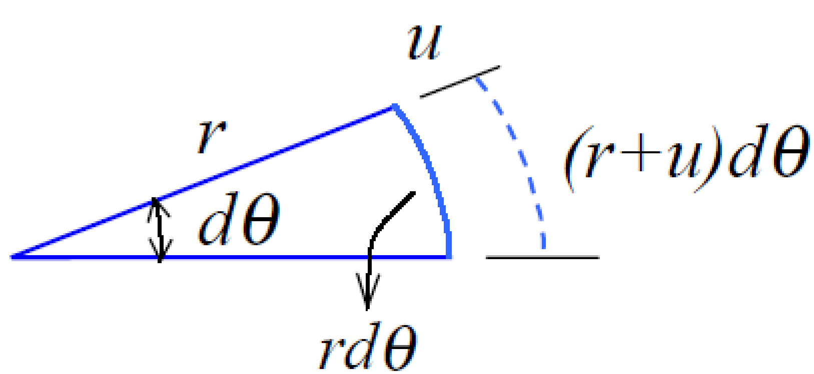
Because of symmetry the stress components are independent of the angular co-ordinate **'θ'** and hence all the derivatives with respect to 'θ ' vanish and the components $\gamma_{x\theta}$, $\gamma_{r\theta}$ are zero. The strain displacement relation are given by

$$\varepsilon_r = \frac{\partial u}{\partial x} \quad ; \quad \varepsilon_\theta = \frac{u}{r} \quad ; \quad \varepsilon_z = \frac{\partial w}{\partial z} \quad ; \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

Strains:

$$\varepsilon_r = \frac{\partial u}{\partial r}\ , \qquad \varepsilon_\theta = \frac{u}{r}\ , \qquad \varepsilon_z = \frac{\partial w}{\partial z}\ ,$$

$$\gamma_{rz} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\ , \ (\gamma_{r\theta} = \gamma_{z\theta} = 0)$$

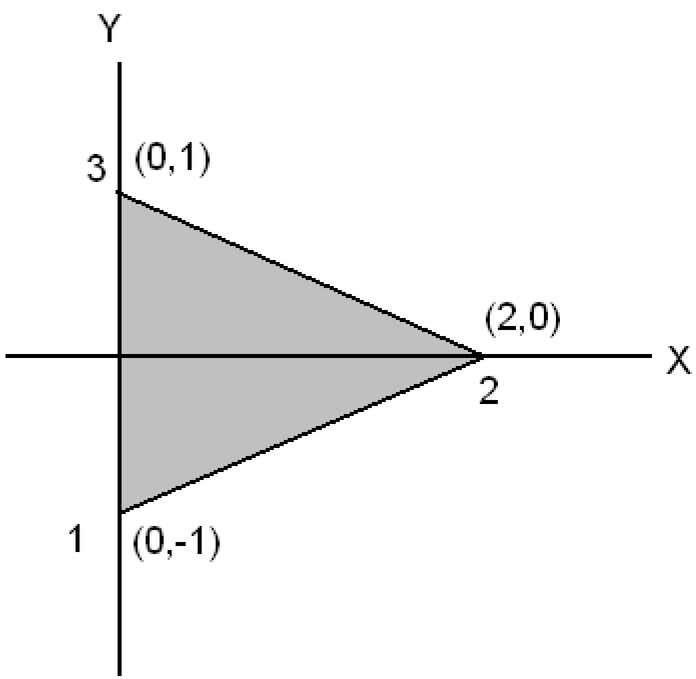


The constitutive relation is

Stresses:

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \dfrac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{Bmatrix}$$

Problem 2:- Assuming plane stress conditions evaluate the stiffness matrix for the element shown in Fig. Assume E= 2 x 10^5 N/cm^2 and μ=0.3. u_1=0.000, v_1=0.0025, u_2=0.0012, v_2=0.000, u_3 =0.0000 & v_3= 0.0025.



$\beta_1 = y_2 - y_3 \quad = 0 - 1 = -1$

$\beta_2 = y_3 - y_1 \quad = 1 + 1 = \ 2$

$\beta_3 = y_1 - y2 \quad = -1 - 0 = -1$

$\gamma_1 = -(x_2 - x_3) \ = 0 - 2 = -2$

$\gamma_2 = -(x_3 - x_1) \ = 0 - 0 = \ 0$

$\gamma_3 = -(x_1 - x_2) \ = 2 - 0 = \ 2$

. $A = \frac{1}{2} \times b \times h = \frac{1}{2} \times 2 \times 2 = 2$

$$\{\epsilon\} = \frac{1}{2A}\begin{pmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{pmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$= \{B]\ \{d\}$$

$$[B] = \frac{1}{2(2)} \begin{pmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{pmatrix}$$

$$[D] = \frac{E}{1-\mu^2}\begin{pmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{pmatrix}$$

$$= \frac{2 \text{ x } 10^5}{1-(0.3)^2}\begin{pmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{pmatrix}$$

Now we know that the stiffness matrix [K] is given by $\int_{vol} [B]^T [D] [B] dv$

Assuming unit thickness ie t = 1 we get

$[K] = A\, [B]^T [D] \{B]$

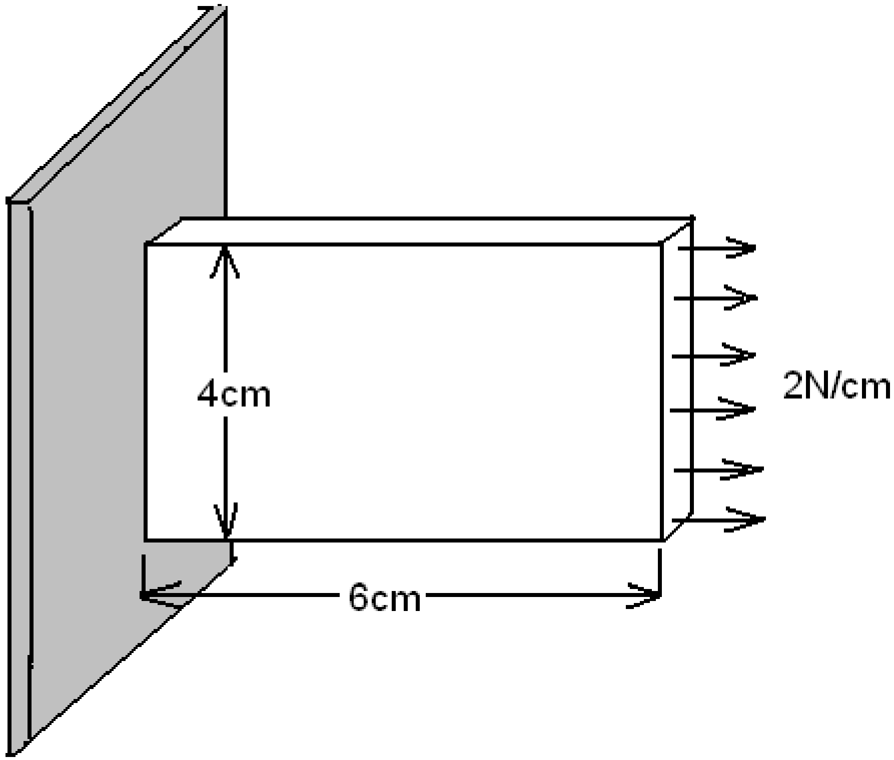
$$= \frac{(2)\,(2 \times 10^5)}{4\,(0.91)} \begin{pmatrix} -1 & 0 & -2 \\ 0 & -2 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 1 & 0.35 \end{pmatrix} \frac{1}{4} \begin{pmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{pmatrix}$$

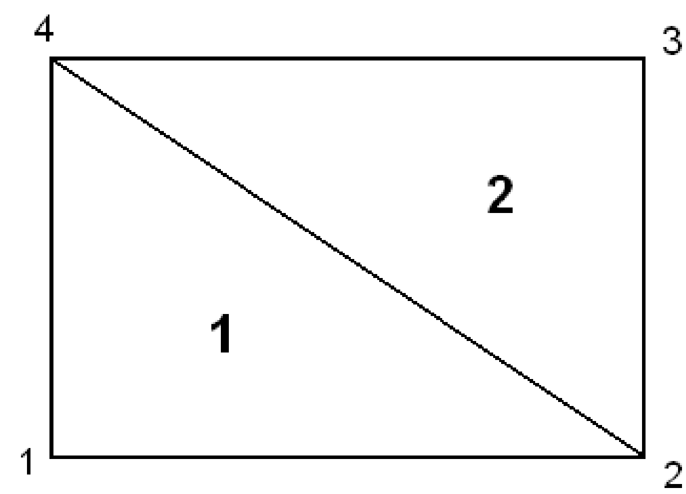
$$\qquad\qquad 6 \times 3 \qquad\qquad 3 \times 3 \qquad\qquad 3 \times 6$$

$$= 1.099 \times 10^2 \begin{pmatrix} 600 & 325 & -500 & -350 & -100 & 25 \\ 325 & 1087.5 & -300 & -175 & -25 & -912.5 \\ -500 & -300 & 1000 & 0 & -500 & 300 \\ -350 & -175 & 0 & 350 & 350 & -175 \\ -100 & -25 & -500 & 350 & 600 & -325 \\ 25 & -912.5 & 300 & -175 & -325 & 1087.5 \end{pmatrix}$$

6 x 6

Note: In order to evaluate the element stress we can use the equation $\{\sigma\} = [D]\ [B]\ \{d\}$



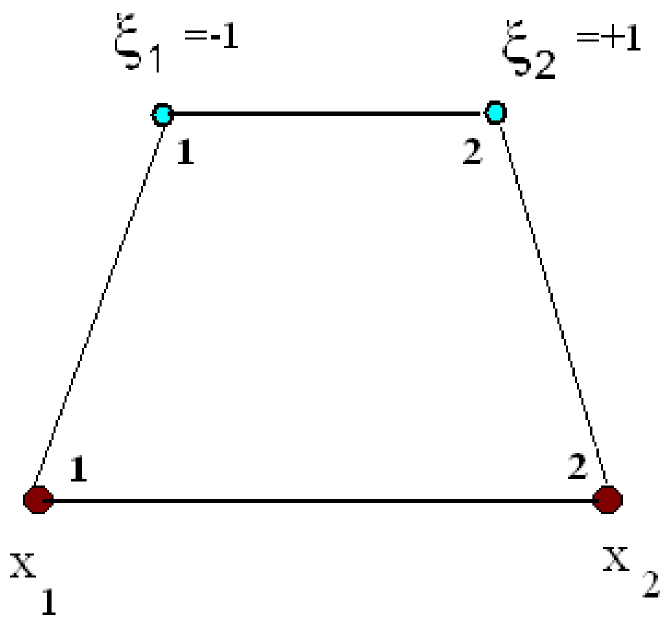


$$F_{3x} = F_{4x} = (2*4)/2 = 4N$$

BC: $u_1 = v_1 = u_4 = v_4 = 0$

NATURAL CO-ORDINATE SYSTEMS

A Natural Co-ordinate system is a local co-ordinate system that permits the specification of a point within an element by a set of dimensionless numbers whose absolute magnitude never exceeds unity i.e. A I Dimensional element described by means of its two end vertices (x_1 & x_2) in Cartesian space is represented or mapped on to Natural co-ordinate space by the line whose end vertices ξ_1 & ξ_2 are given by –1 & +1 respectively.



ADVANTAGES OF NATURAL CO-ORDINATE SYSTEMS

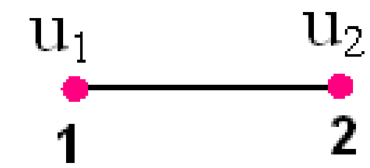
i) It is very convenient in constructing interpolation functions.

ii) Integration involving Natural co-ordinate can be easily performed as the limits of the Integration is always from –1 to +1. This is in contrast to global co-ordinates where the limits of Integration may vary with the length of the element.

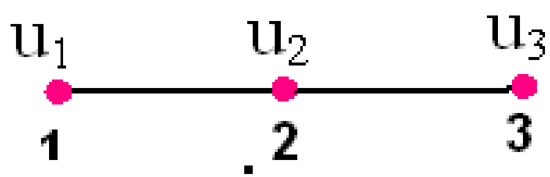
iii) The nodal values of the co-ordinates are convenient number or fractions.

iv) It is possible to have elements with curved sides.

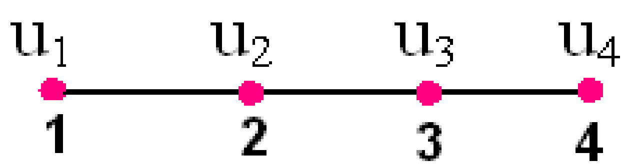
I D elements



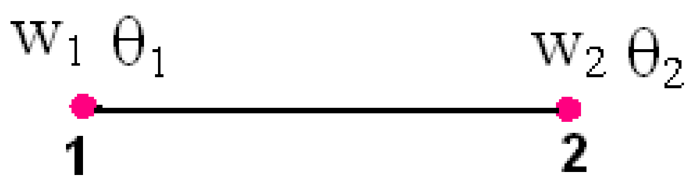
2 NODED LINEAR ELEMENT



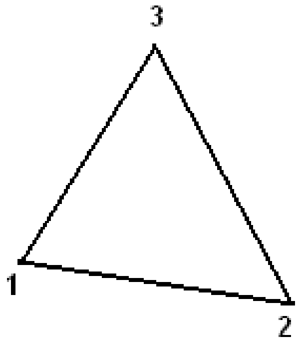
3 NODED QUADRATIC ELEMENT



4 NODED CUBIC ELEMENT

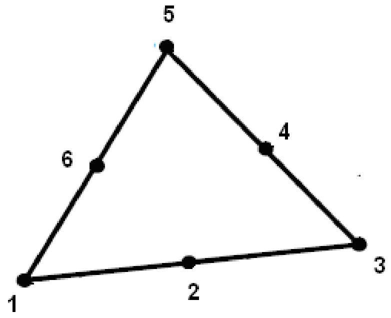


2 NODED BEAM ELEMENT

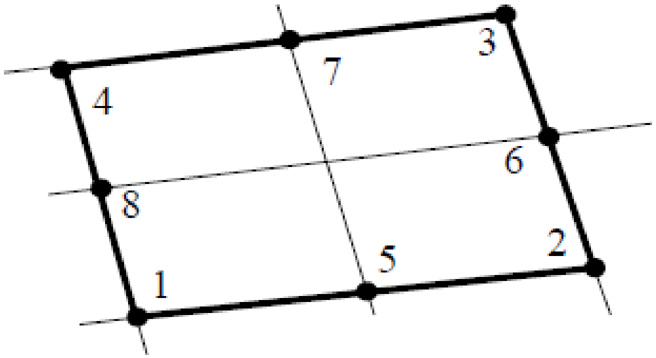


Constant strain triangular element

Bilinear Rectangular element

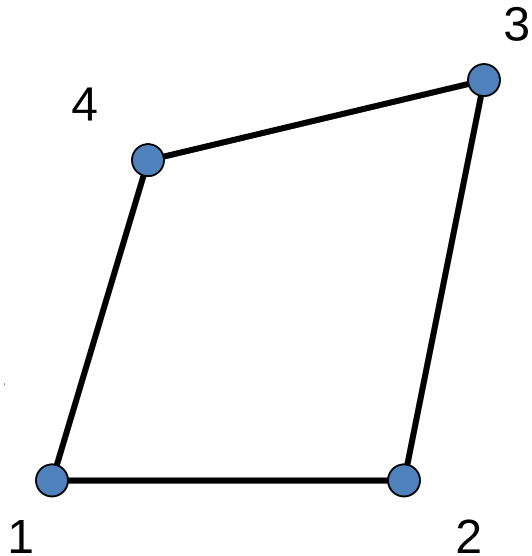


Linear strain triangular element



Eight noded quadratic quadrilateral elemen

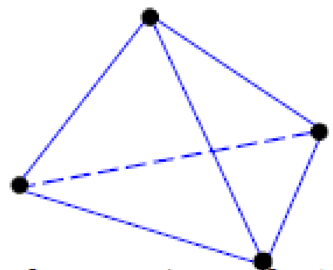
II D elements



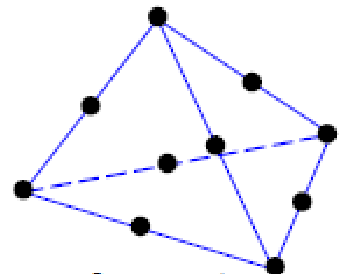
Linear Quadrilateral element

III D elements

Tetrahedron:

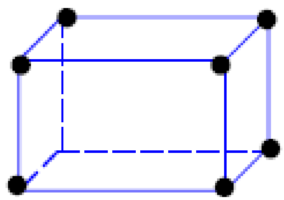


linear (4 nodes)

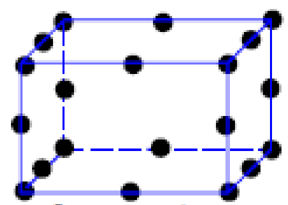


quadratic (10 nodes)

Hexahedron (brick):

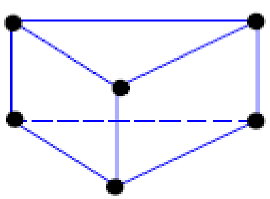


linear (8 nodes)

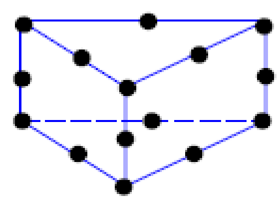


quadratic (20 nodes)

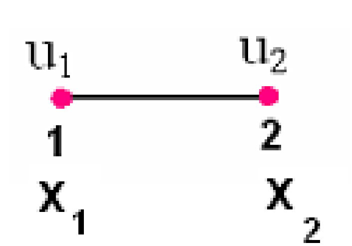
Penta:



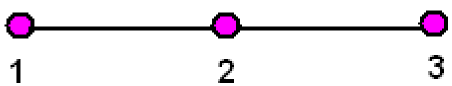
linear (6 nodes)

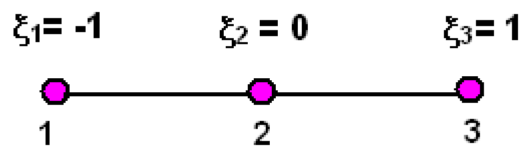


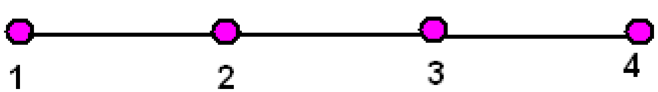
quadratic (15 nodes)

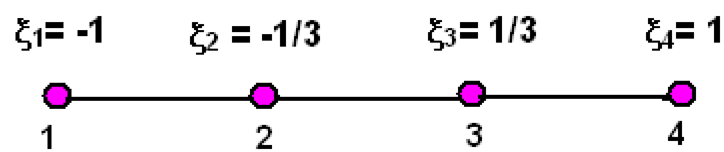


2 NODED LINEAR ELEMENT



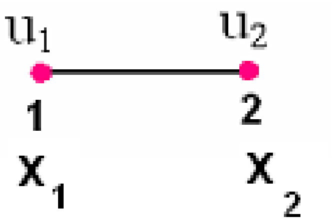


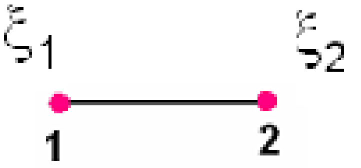




I - D Lagrangian Interpolation functions in Natural Co-ordinates

Linear Element:





$$L_1 = \frac{(\xi - \xi_2)}{(\xi_1 - \xi_2)}$$

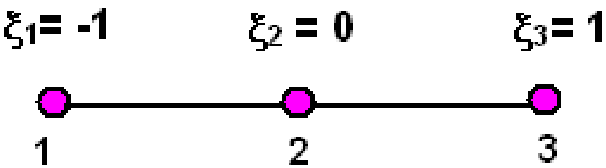
Substituting $\xi_1 = -1$ & $\xi_2 = +1$, we get

$$L_1 = \frac{(\xi - 1)}{-1\ -1} = \frac{1 - \xi}{2} = \frac{1}{2}(1 - \xi)$$

$$L_2 = \frac{(\xi - \xi_1)}{(\xi_2 - \xi_1)} = \frac{(\xi + 1)}{+1\ +1} = \frac{1}{2}(1 + \xi)$$

In general $L_j = ½ (1 + \xi\xi_j)$

3 Noded Quadratic Element



$$\xi_1 = -1 \quad \xi_2 = 0 \quad \xi_3 = 1$$

$$L_1 = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} = \frac{(\xi - 0)(\xi - 1)}{(-1 - 0)\ (-1 - 1)} = \xi/2\ (\xi - 1)$$

$$= -\xi/2\ (1 - \xi)$$

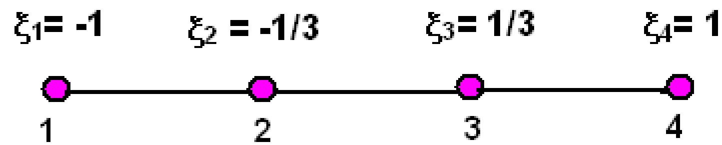
$$L_2 = \frac{(\xi - \xi_1)(\xi - \xi_3)}{(\xi_2 - \xi_1)\ (\xi_2 - \xi_3)} = \frac{(\xi + 1)(\xi - 1)}{(0 + 1)\quad (0 - 1)} = (1 - \xi)\ (1 + \xi)$$

$$L_3 = \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_3 - \xi_1)\ (\xi_3 - \xi_2)} = \frac{(\xi + 1)(\xi - 0)}{(1 + 1)\ \ (1 - 0)} = \xi/2\ (1 + \xi)$$

4 Noded Cubic Element:

$\xi_1 = -1 \quad \xi_2 = \frac{1}{3} \quad \xi_3 = \frac{1}{3} \quad \xi_4 = 1$

$$L1 = \frac{(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)(\xi_1 - \xi_4)}$$



$$= \frac{(\xi + \frac{1}{3})(\xi - \frac{1}{3})(\xi - 1)}{(-1 + \frac{1}{3})(-1 - \frac{1}{3})(-1 - 1)} = -9/16\, (\tfrac{1}{3} + \xi)(1 - \xi)(\tfrac{1}{3} - \xi)$$

$$L2 = \frac{(\xi - \xi_3)(\xi - \xi_4)(\xi - \xi_1)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)(\xi_2 - \xi_4)}$$

$$= -27/16\, (1 + \xi)\, (1 - \xi)\, (\tfrac{1}{3} - \xi)$$

$$L_3 = \frac{(\xi - \xi_1)(\xi_1 - \xi_2)(\xi - \xi_4)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)(\xi_3 - \xi_4)}$$

$$= 27/16\ (1+\xi)(1 - \xi)(1/3 + \xi)$$

$$L_4 = \frac{(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)}{(\xi_4 - \xi_1)(\xi_4 - \xi_2)(\xi_4 - \xi_3)}$$

$$= -9/16\ (1/3+\xi)(1/3 - \xi)(1 + \xi)$$

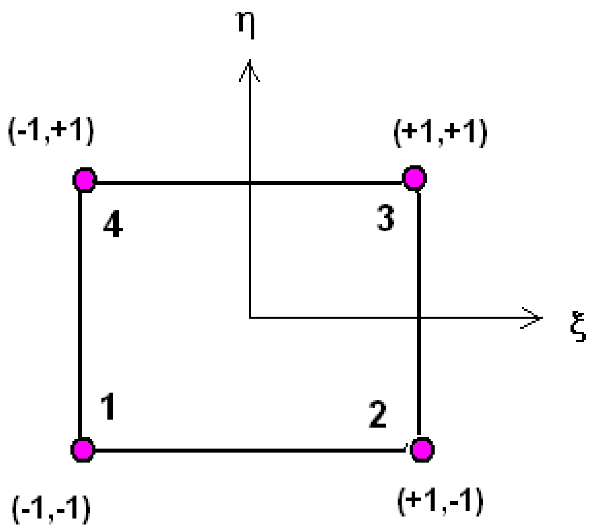
Lagrangian Interpolation polynomials for rectangular Element: (Natural Co-ordinates)

Bi-Linear rectangular Element:

$$N_1(\xi) = \frac{\xi - \xi_2}{\xi_1 - \xi_2} = \frac{\xi - 1}{-1 - 1} = \frac{1 - \xi}{2}$$

$$N_1(\eta) = \frac{\eta - \eta_4}{\eta_1 - \eta_4} = \frac{\eta - 1}{-1 - 1} = \frac{1 - \eta}{2}$$

$$\therefore N_1(\xi,\eta) = N_1(\xi)\, N_1(\eta)$$



$$= \left(\frac{1 - \xi}{2}\right)\left(\frac{1 - \eta}{2}\right)$$

$$= 1/4(1 - \xi)(1 - \eta)$$

$$N_2(\xi,\eta) = \frac{(\xi - \xi_1)(\eta - \eta_3)}{(\xi_2 - \xi_1)(\eta_2 - \eta_3)}$$

$$= \frac{(\xi + 1)(\eta - 1)}{(1 + 1)(-1 - 1)} \qquad = \tfrac{1}{4}(1 + \xi)(1 - \eta)$$

$$N_3(\xi,\eta) = \frac{(\xi - \xi_4)(\eta - \eta_2)}{(\xi_3 - \xi_4)(\eta_3 - \eta_2)}$$

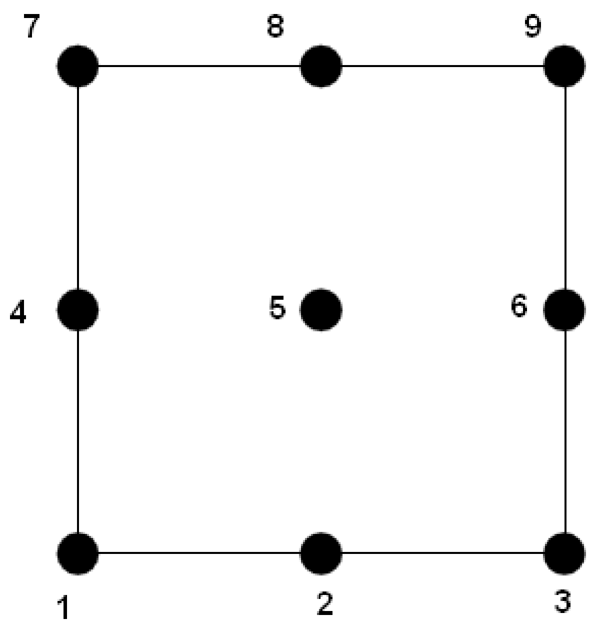
$$= \frac{(\xi + 1)(\eta + 1)}{(1 + 1)(1 + 1)} \qquad = \tfrac{1}{4}(1 + \xi)(1 + \eta)$$

$$N_4(\xi,\eta) = \frac{(\xi - \xi_3)(\eta - \eta_1)}{(\xi_4 - \xi_3)(\eta_4 - \eta_1)}$$

$$= \frac{(\xi - 1)(\eta + 1)}{(-1 - 1)(1 + 1)}$$

$$= \tfrac{1}{4}(1 - \xi)(1 + \eta)$$

NINE NODED QUADRATIC QUADRILATERAL ELEMENT



We shall now proceed to derive the shape functions for a nine noded quadratic quadrilateral element using Lagrangian polynomials, in natural co-ordinates.

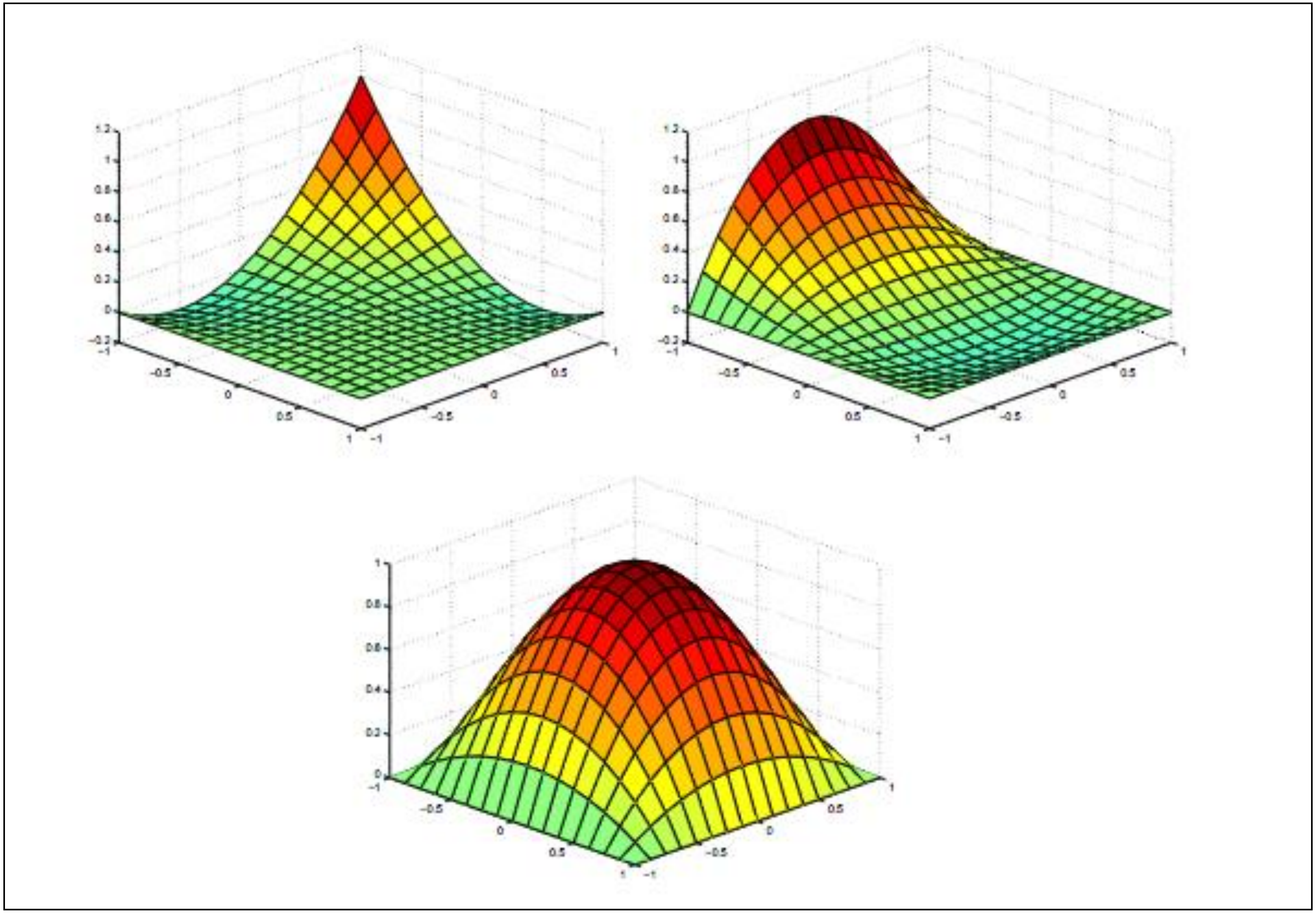
$$N_1(\xi) = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)}$$

$$= \frac{(\xi - 0)(\xi - 1)}{(-1 - 0)(-1 - 1)} = \frac{\xi(\xi - 1)}{2}$$

$$N_1(\eta) = \frac{(\eta - \eta_4)(\eta - \eta_7)}{(\eta_1 - \eta_4)(\eta_1 - \eta_7)}$$

$$= \frac{(\eta - 0)(\eta - 1)}{(-1 - 0)(-1 - 1)} = \frac{\eta(\eta - 1)}{2}$$

$$\therefore N_1(\xi,\eta) = N_1(\xi)\, N_2(\eta) = \tfrac{1}{4}(\xi^2 - \xi)(\eta^2 - \eta)$$



$$N_2(\xi,\eta) = \tfrac{1}{2}\,(1 - \xi^2)\,(\eta^2 - \eta)$$

$$N_3(\xi,\eta) = \tfrac{1}{4}\,(\xi^2 + \xi)\,(\eta^2 - \eta)$$

$$N_4(\xi,\eta) = \tfrac{1}{2}\,(\xi^2 - \xi)\,(1 - \eta^2)$$

$$N_5(\xi,\eta) = (1 - \xi^2)\,(1 - \eta^2)$$

$$N_6(\xi,\eta) = \tfrac{1}{2}\,(\xi^2 + \xi)\,(1 - \eta^2)$$

$$N_7(\xi,\eta) = \tfrac{1}{4}\,(\xi^2 - \xi)\,(\eta^2 + \eta)$$

$$N_8(\xi,\eta) = \tfrac{1}{2}\,(1 - \xi^2)\,(\eta^2 + \eta)$$

$$N_9(\xi,\eta) = \tfrac{1}{4}\,(\xi^2 + \xi)\,(\eta^2 + \eta)$$

Shape functions for Eight noded quadrilateral element :

The equations to the various lines connecting the various nodes is given by

Line 1 – 2 – 3 $\longrightarrow$ $1 + \eta = 0$

Line 6 – 7 – 8 $\longrightarrow$ $1 - \eta = 0$

Line 1 – 4 – 6 $\longrightarrow$ $1 + \xi = 0$

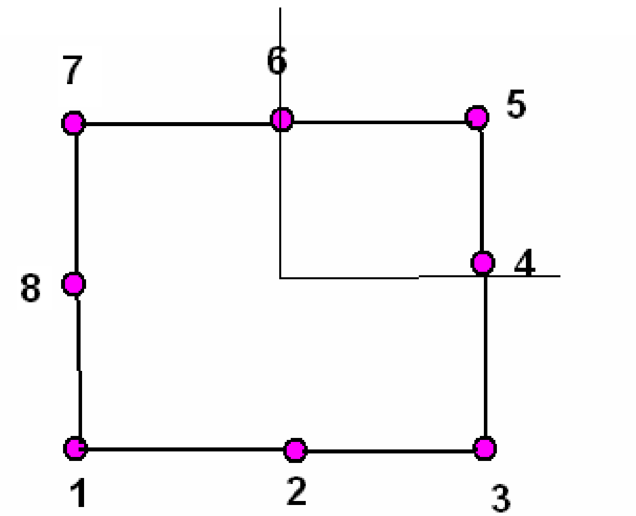
Line 3 – 5 – 8 $\longrightarrow$ $1 - \xi = 0$

Line 2 – 5 $\longrightarrow$ $1 - \xi + \eta = 0$

Line 4 – 7 $\longrightarrow$ $1 + \xi - \eta = 0$

Line 7 – 5 $\longrightarrow$ $1 - \xi - \eta = 0$

Line 4 – 2 $\longrightarrow$ $1 + \xi + \eta = 0$



To obtain the shape function N_1 , we identify the equation to those lines not passing through node 1 and express N_1 as a product of these line equations.

i.e. lines 6 – 7 – 8, 3 – 5 – 8 and 4 – 2

$$\therefore N_1 = C(1 - \eta)\,(1 - \xi)\,(1 + \xi + \eta)$$

$$\therefore N_1\,(-1, -1) = C(1 + 1)\,(1 + 1)\,(1 - 1 - 1) = 1$$

$$\therefore C = -1/4$$

$$\therefore N_1(\xi,\eta) = -\tfrac{1}{4}\,(1 - \eta)(1 - \xi)\,(1 + \xi + \eta)$$

Similarly for N_2 the lines are 6 – 7 – 8 , 1 – 4 – 6 and 3 – 5 – 8

$$\therefore N_2 = C(1 - \eta)\,(1 + \xi)\,(1 - \xi)$$
$$= C(1 - \eta)\,(1 - \xi^2)$$

$$N_2\ (0, -1) = C(1 – 0)\,(1 + 1) = 1$$
$$\therefore C = ½$$

$$\therefore N_2\,(\xi,\eta) = ½\,(1 – \xi^2)(1 - \eta)$$

$$N_3\,(\xi,\eta) = ¼\,(1 + \xi)\,(1 - \eta)\,(-1 + \xi - \eta)$$

$$N_4(\xi,\eta) = \tfrac{1}{2} (1 - \xi)(1 - \eta^2)$$

$$N_5 (\xi,\eta) = \tfrac{1}{2} (1 + \xi)(1 - \eta^2)$$

$$N_6 (\xi,\eta) = \tfrac{1}{4} (1 - \xi)(1 + \eta)(-1 - \xi + \eta)$$

$$N_7 (\xi,\eta) = \tfrac{1}{2} (1 - \xi^2)(1 + \eta)$$

$$N_8 (\xi,\eta) = \tfrac{1}{4} (1 + \xi)(1 + \eta)(-1 + \xi + \eta)$$

ISOPARAMETRIC ELEMENTS

$$x = \sum_{i=1}^{r} x_i L_i(\xi)$$

For a linear transformation r = 2

$$\therefore x = x_1 L_1(\xi) + x_2 L_2(\xi)$$

$$= \frac{x_1(1-\xi)}{2} + \frac{x_2(1+\xi)}{2}$$

For example an element whose x co-ordinates are given by $x_1 = 3$ & $x_2 = 7$

Then $x_1 = x_1 \dfrac{(1 - \xi)}{2} + x_2 \dfrac{(1 + \xi)}{2}$

$$3 = \frac{3(1 - \xi)}{2} + \frac{7(1 + \xi)}{2}$$

or $6 = 3 - 3\xi + 7 + 7\xi$

or $4\xi = -4$

or $\xi = -1$

ie the point $x_i = 3$ transforms to $\xi = -1$ in natural co-ordinate space similarly $x_2 = x_1 \dfrac{(1 - \xi)}{2} + x_2 \dfrac{(1 + \xi)}{2}$

$$7 = 3 \frac{(1 - \xi)}{2} + 7 \frac{(1 + \xi)}{2}$$

$$14 = 3 - 3\xi + 7 + 7\xi$$

$$4\xi = 4 \quad \text{or} \quad \xi = 1$$

$\therefore$ The point $x_2 = 7$ in Cartesian space gets transformed to $\xi_2 = +1$ in Natural co-ordinate space. So the transformation

$$X = \sum_{i=1}^{r} {}_i \; \alpha_i \; (\xi)$$ transforms the geometry

Similarly we have the approximation of the field variable in terms of shape functions expressed as

$$u = \sum_{i=1}^{s} u_i N_i(\xi)$$

Jacobian of Transformation

Among the 3 cases given above Isoparametric are more commonly used due to their advantages which include the following:

i) Quadrilateral elements in (x,y) coordinates with curved boundaries get transformed to a rectangle of (2 x 2) units in (ξ, η) co-ordinates

ii) Numerical integration is more easily performed as limits of integration vary from –1 to +1 for all elements.

We have seen that determination of the stiffness matrix requires the computation of derivative of shape functions with respect to ‘x’. However as the shape functions (Interpolation function) are expressed in terms of ξ & η co-ordinates (natural co-ordinates) we use the chain rule.

$$\frac{dN_1}{dx} = \frac{dN_1}{d\xi} \frac{d\xi}{dx} = \frac{dN_1}{d\xi} \frac{1}{dx / d\xi}$$

$$= \frac{dN_1}{d\xi} \frac{1}{J} = J\text{-}1 \frac{dN_1}{d\xi}$$

Here $J = dx/ d\xi$ is the 'Jacobian' of transformations from Cartesian space to natural co-ordinate space. It can be considered as the scale factor between the two co-ordinate systems.

Jacobian of transformation for 2 Noded Linear Element

For a 2 Noded element the shape functions are given by

$$N_1(\xi) = \frac{(1 - \xi)}{2}$$

$$N_2(\xi) = \frac{(1+\xi)}{2}$$

Now $x = N_1 x_1 + N_2 x_2$

$$= \frac{(1 - \xi)}{2} x_1 + \frac{(1 + \xi)}{2} x_2$$

$$\frac{dx}{d\xi} = J = \frac{-1}{2} x_1 + \frac{1}{2} x_2$$

$$= \frac{(x_2 - x_1)}{2} = \frac{L}{2}$$

Here $(x_2 - x_1)$ represents the length of the element. So the Jacobian of transformation for a 2 noded element is given by L/2

3- Noded Quadratic element:-

$N_1 = -\xi/2\ (1- \xi)$

$N_2 = (1- \xi)\ (1+ \xi)$

$N_3 = \ \xi/2\ (1+ \xi)$

$u = N_1\ u_1 + N_2\ u_2 + N_3\ u_3$ &

$x = N_1\ x_1\ N_2\ x_2 + N_3\ x_3$

$$J = \frac{dx}{d\xi} = \begin{pmatrix} \frac{-1 + 2\xi}{2} & -2\xi & \frac{1 + 2\xi}{2} \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Stiffness Matrix for a 2 Noded Axial Element

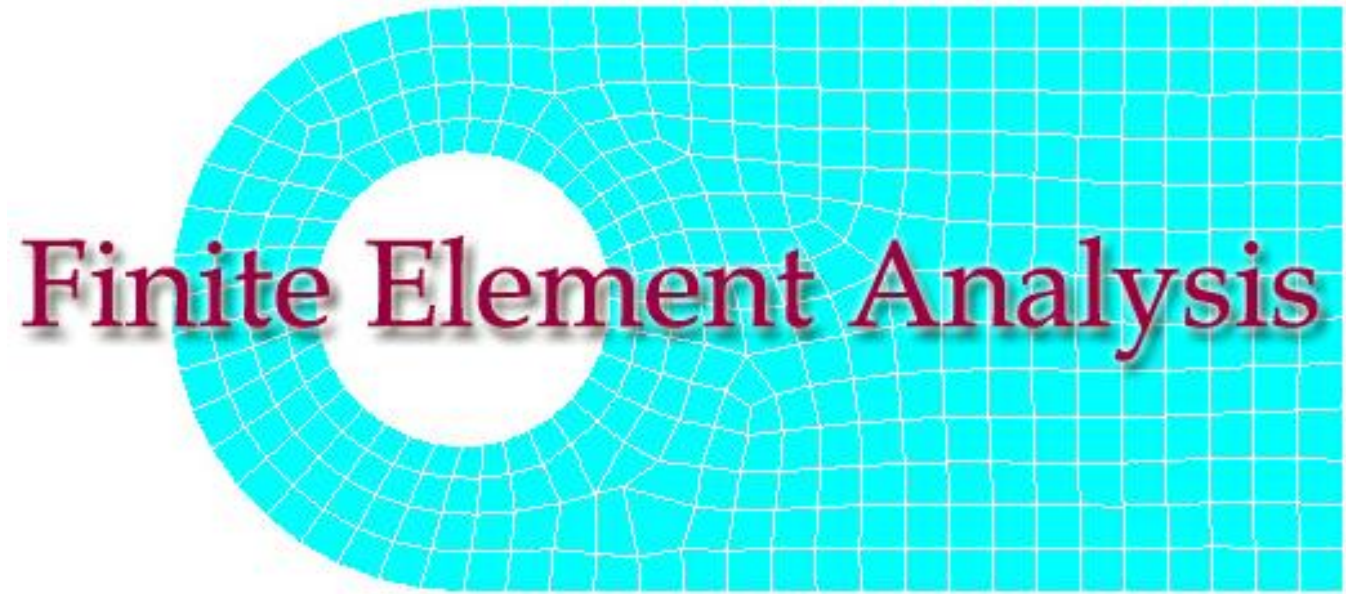
$$[K] = \int_0 B^T D\, BA\, dx$$

$$[B] = \frac{du}{dx} = \frac{dN}{dx} = \frac{1}{J}\frac{dN}{d\xi}$$

$$= \frac{2}{L}\begin{pmatrix} \frac{dN_1}{d\xi} & \frac{dN_2}{d\xi} \end{pmatrix}$$

$$= \frac{2}{L}\begin{pmatrix} \frac{d}{d\xi}\frac{(1-\xi)}{2} & \frac{d}{d\xi}\frac{(1+\xi)}{2} \end{pmatrix}$$

$$= \frac{2}{L}\begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{-1}{L} & \frac{1}{L} \end{pmatrix}$$



ISOPARAMETRIC TRANSFORMATION
and
NUMERICAL INTEGRATION

LECTURE 12

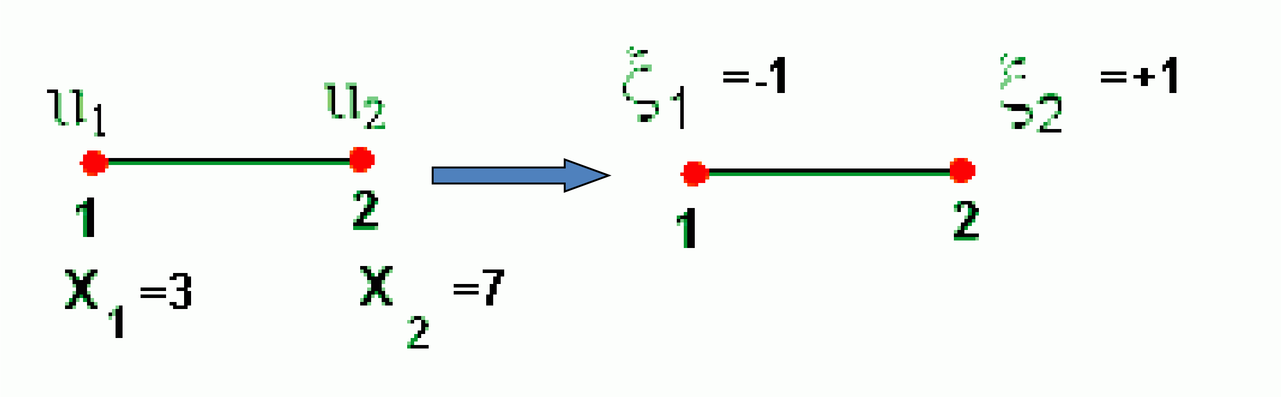
ISOPARAMETRIC ELEMENTS

$$x = \sum_{i=1}^{r} x_i L_i(\xi)$$

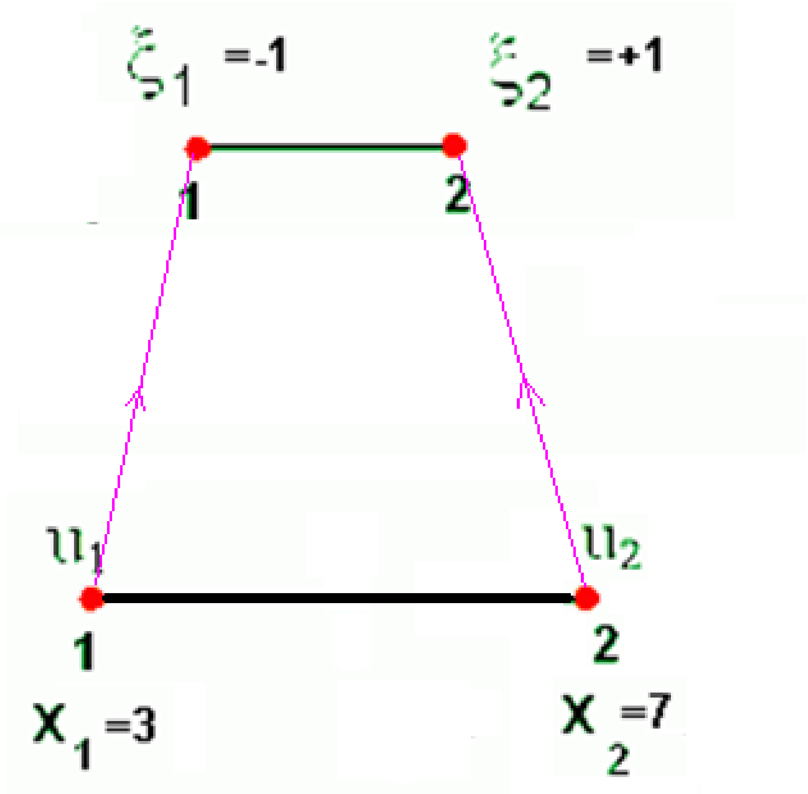
For a linear transformation r = 2

$$\therefore x = x_1 N_1(\xi) + x_2 N_2(\xi)$$

$$= x_1 \frac{(1-\xi)}{2} + x_2 \frac{(1+\xi)}{2}$$



u_1
u_2
1
2
X_1 =3
X_2 =7
ξ_1 =-1
ξ_2 =+1
1
2



ξ_1 =-1
ξ_2 =+1
1
2
u_1
u_2
1
2
X_1 =3
X_2 =7

For example an element whose x co-ordinates are given by $x_1 = 3$ & $x_2 = 7$

Then $$x_1 = x_1 \frac{(1 - \xi)}{2} + x_2 \frac{(1 + \xi)}{2}$$

$$3 = \frac{3(1 - \xi)}{2} + \frac{7(1 + \xi)}{2}$$

or $6 = 3 - 3\xi + 7 + 7\xi$

or $4\xi = -4$

or $\xi = -1$

ie the point $x_i = 3$ transforms to $\xi = -1$ in natural co-ordinate space similarly $x_2 = x_1 \dfrac{(1-\xi)}{2} + x_2 \dfrac{(1+\xi)}{2}$

$$7 = 3\,\frac{(1-\xi)}{2} + 7\,\frac{(1+\xi)}{2}$$

$$14 = 3 - 3\xi + 7 + 7\xi$$

$$4\xi = 4 \quad \text{or} \quad \xi = 1$$

$\therefore$ The point $x_2 = 7$ in Cartesian space gets transformed to $\xi_2 = +1$ in Natural co-ordinate space. Similarly every point in X space transforms to a corresponding point in ξ space

So the transformation

$$X = \sum_{i=1}^{r} N_i \, x_i \, (\xi)$$ transforms the geometry

from Cartesian space to Gaussian space

Similarly we have the approximation of the field variable in terms of shape functions expressed as

$$u = \sum_{i=1}^{s} u_i \, N_i \, (\xi)$$

Here **‘r’** - the number of nodes used for geometric transformation

‘s’ - the number of nodes used for approximation of field variable.

In general the polynomial used for geometric transformation need not be of the same order as that used for the field variable approximation.

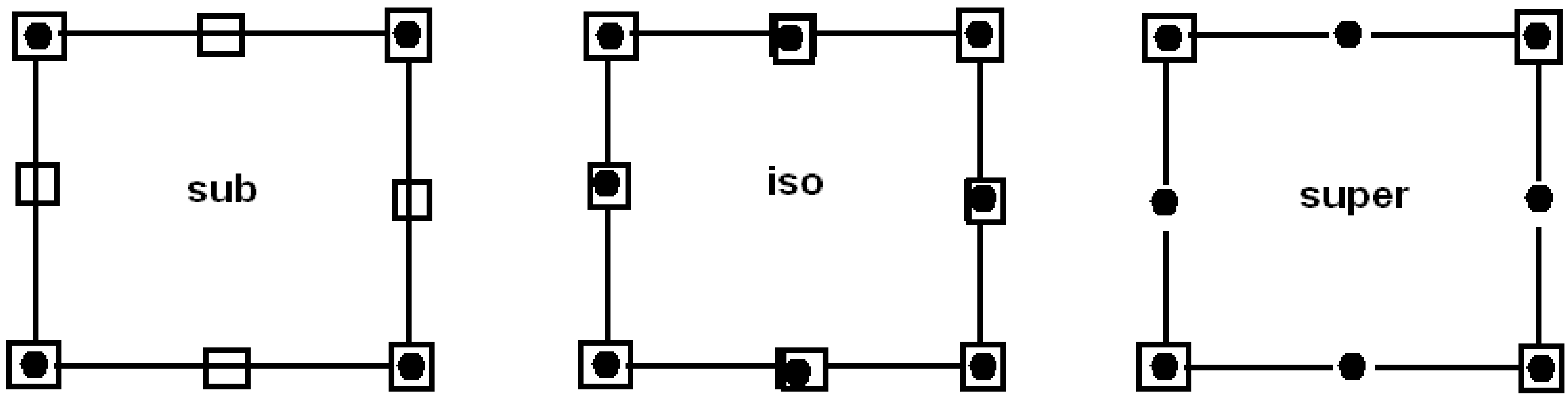
In other words two sets of nodes exists for the same region or element.

- One set of nodes for co-ordinate transformation from Cartesian space to natural co-ordinate space

- One set of nodes for approximating the variation of the field variable over the element.

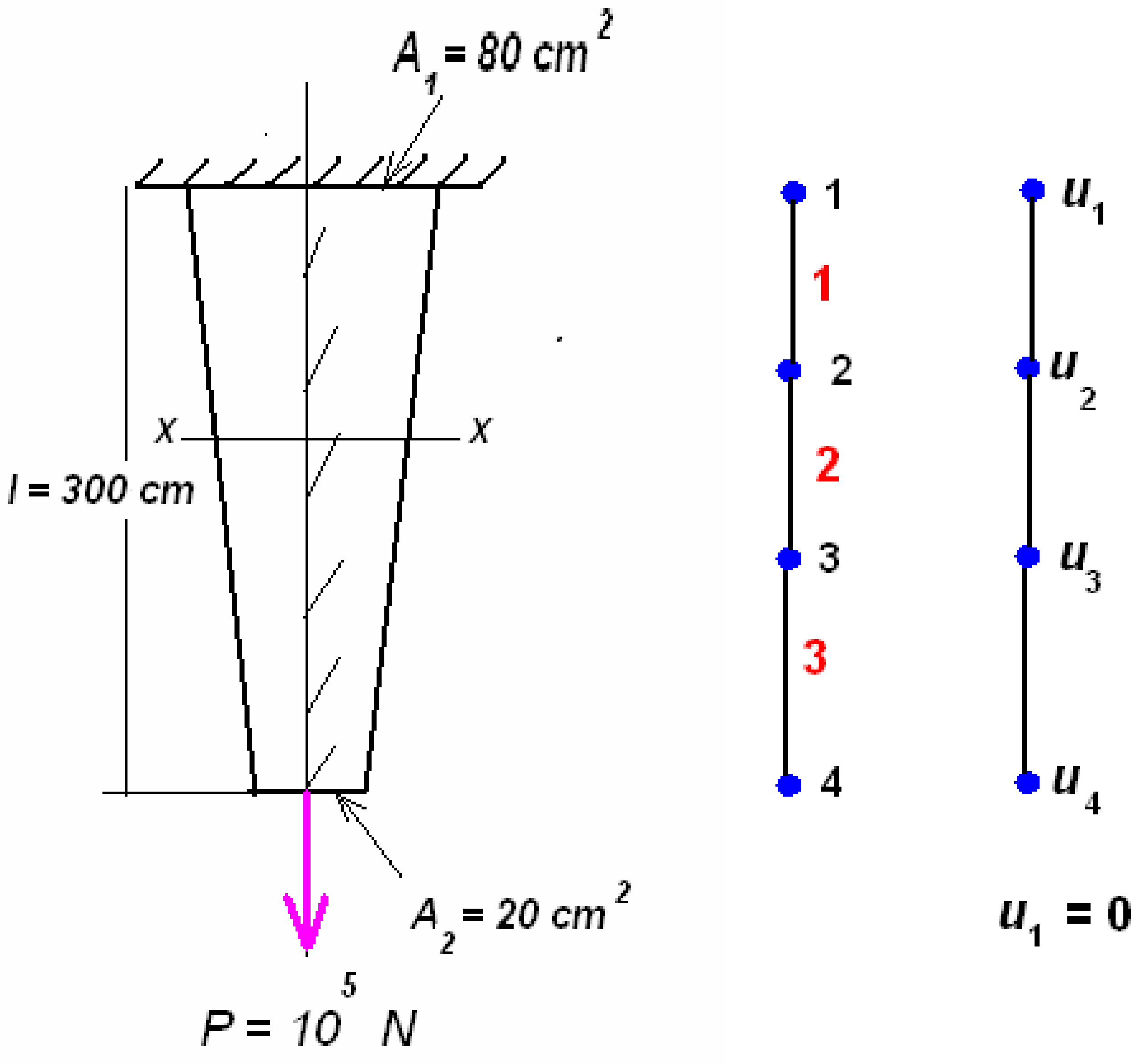
Depending upon the relationship between these two polynomials elements are classified into three categories as

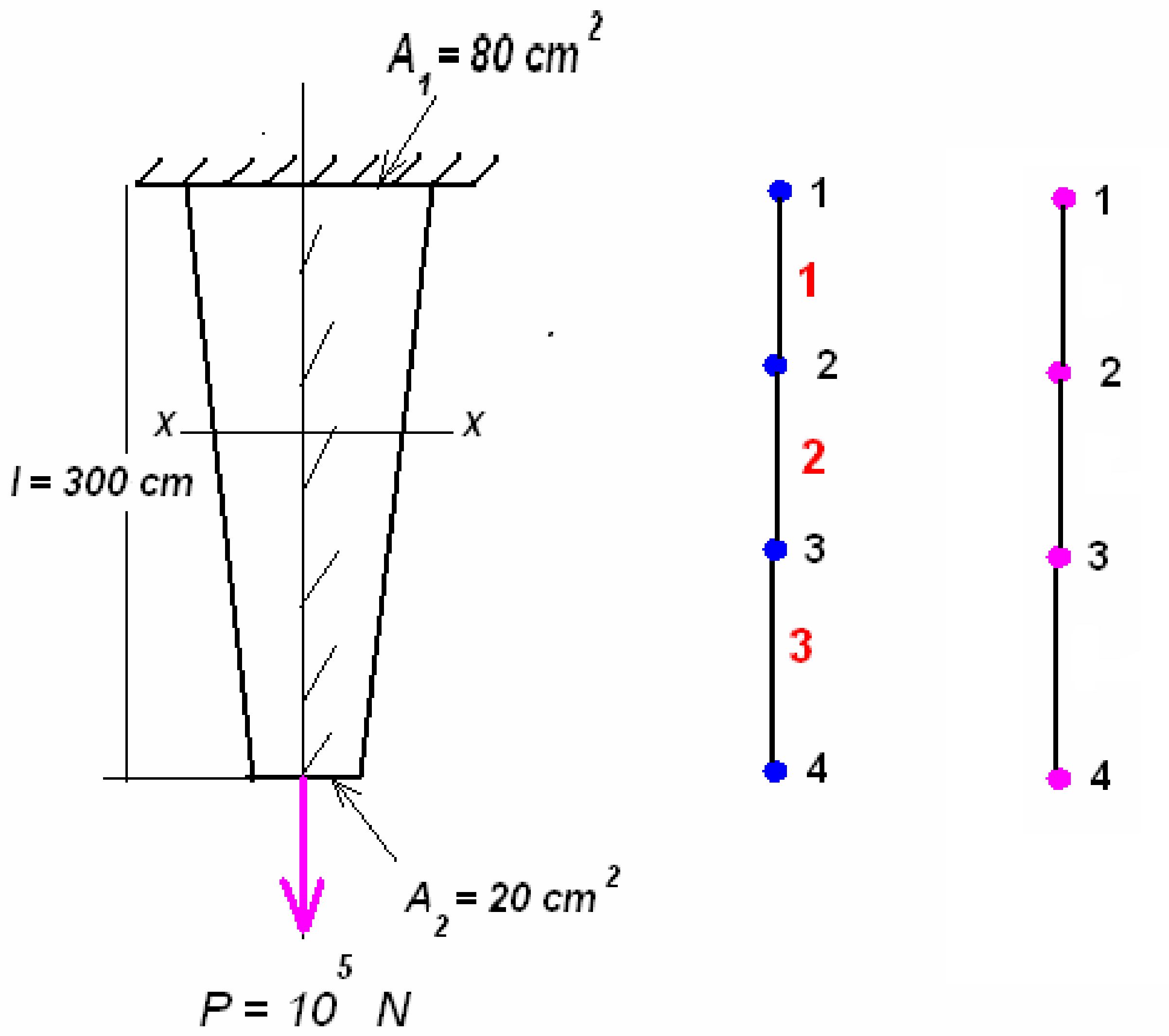
- sub parametric elements $r < s$
- iso-paramatric elements $r = s$
- super-parametric elements $r > s$



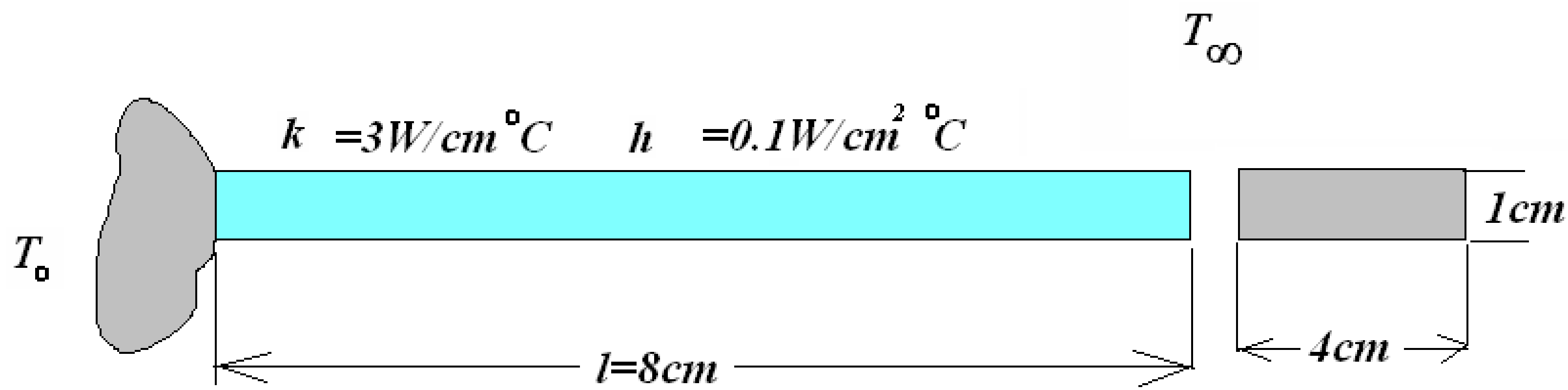
r- nodes for geometric transformation

s- nodes used for field variable approximation

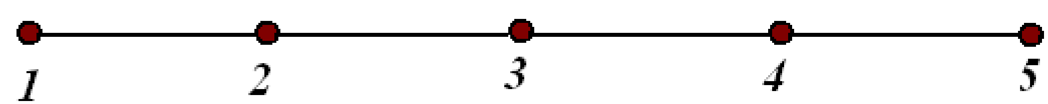

$A_1 = 80\ cm^2$
$l = 300\ cm$
x
x
$A_2 = 20\ cm^2$
$P = 10^5\ N$
1
1
2
2
3
3
4
u_1
u_2
u_3
u_4
$u_1 = 0$



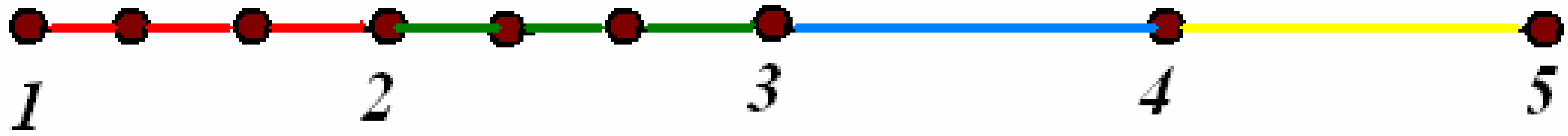
$A_1 = 80\ cm^2$
x
x
$l = 300\ cm$
$A_2 = 20\ cm^2$
$P = 10^5\ N$
1
1
2
2
3
3
4
1
2
3
4



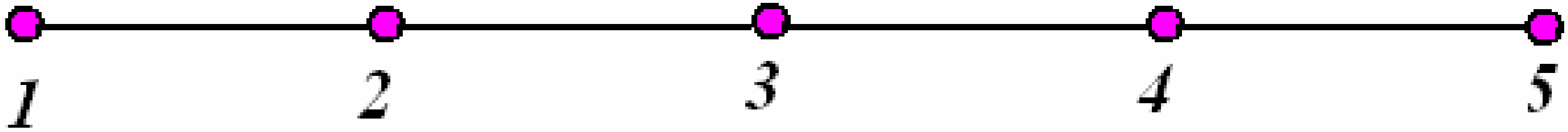
T_∞
k =3W/cm °C
h =0.1W/cm² °C
T_o
1cm
4cm
l=8cm



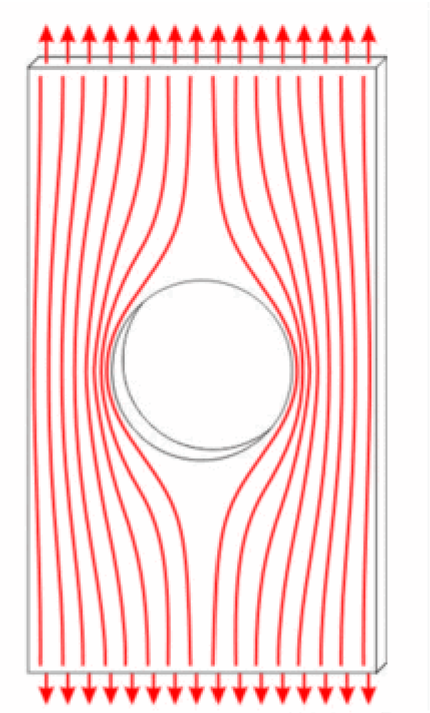
1
2
3
4
5

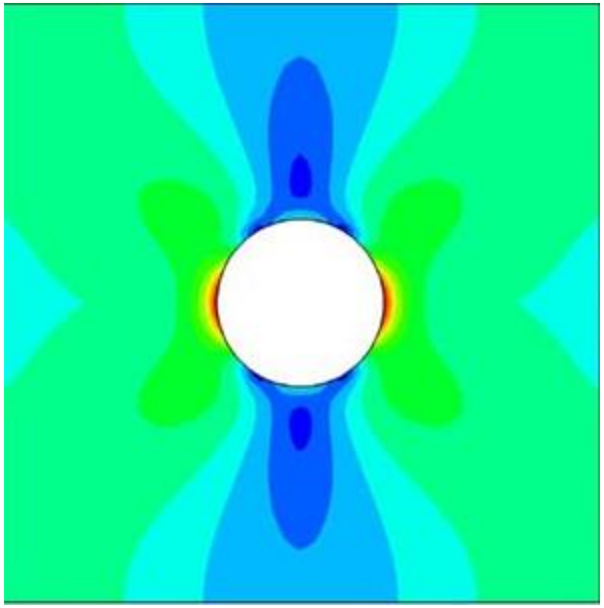


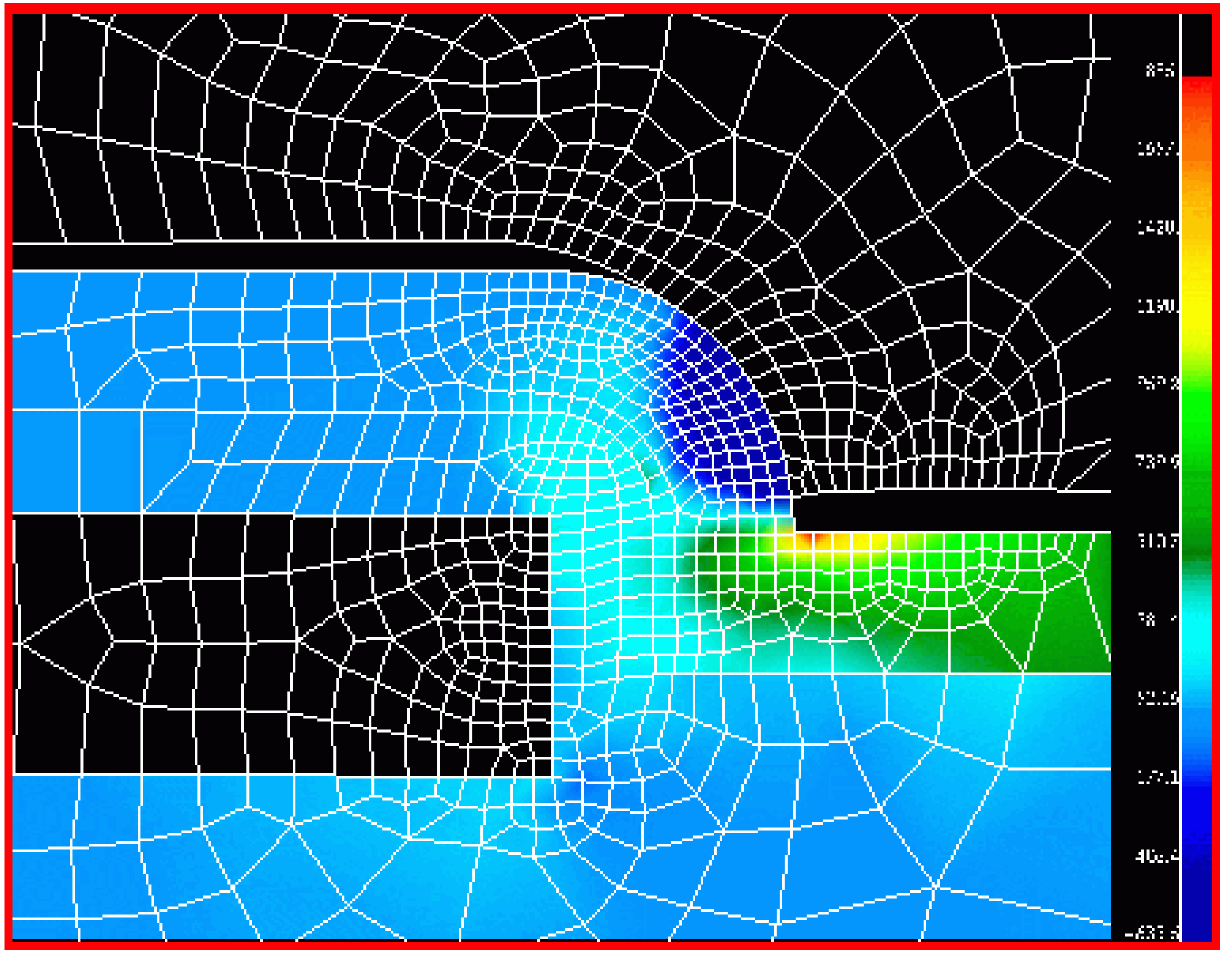
Field variable approximation



Geometric Transformation







Jacobian of Transformation

Among the 3 cases given above Isoparametric are more commonly used due to their advantages which include the following:

i) Quadrilateral elements in (x,y) coordinates with curved boundaries get transformed to a rectangle of (2 x 2) units in (ξ, η) co-ordinates

ii) Numerical integration is more easily performed as limits of integration vary from –1 to +1 for all elements.

We have seen that determination of the stiffness matrix requires the computation of derivative of shape functions with respect to ‘x’. However as the shape functions (Interpolation functions) are expressed in terms of ξ & η co-ordinates (natural co-ordinates) we use the chain rule.

$$\frac{dN_1}{dx} = \frac{dN_1}{d\xi}\frac{d\xi}{dx} = \frac{dN_1}{d\xi}\frac{1}{dx / d\xi}$$

$$= \frac{dN_1}{d\xi}\frac{1}{J} = J^{-1}\frac{dN_1}{d\xi}$$

Here $J = dx/d\xi$ is the 'Jacobian' of transformation from Cartesian space to natural co-ordinate space. It can be considered as the scale factor between the two co-ordinate systems.

Jacobian of transformation for 2 Noded Linear Element

For a 2 Noded element the shape functions are given by

$$N_1(\xi) = \frac{(1 - \xi)}{2}$$

$$N_2(\xi) = \frac{(1+\xi)}{2}$$

$$\text{Now } x = N_1 x_1 + N_2 x_2$$

$$= \frac{(1 - \xi)}{2} x_1 + \frac{(1 + \xi)}{2} x_2$$

$$\frac{dx}{d\xi} = J = \frac{-1}{2} x_1 + \frac{1}{2} x_2$$

$$= \frac{(x_2 - x_1)}{2} = \frac{L}{2}$$

Here $(x_2 - x_1)$ represents the length of the element. So the Jacobian of transformation for a 2 noded element is given by L/2

3- Noded Quadratic element:-

$N_1 = -\xi/2\ (1-\xi)$
$N_2 = (1-\xi)\ (1+\xi)$
$N_3 = \xi/2\ (1+\xi)$
$u = N_1 u_1 + N_2 u_2 + N_3 u_3$ &
$x = N_1 x_1 + N_2 x_2 + N_3 x_3$

$$= -\xi/2(1-\xi)x_1 + (1-\xi)(1+\xi)x_2 + \xi/2(1+\xi)x_3$$

$$J = \frac{dx}{d\xi} = \begin{bmatrix} \frac{-1+2\xi}{2} & -2\xi & \frac{1+2\xi}{2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Jacobian of transformation for 2-D elements:-

In the case of two dimensional elements the shape functions N_i are functions of both x & y. When we obtain the same using Natural co-ordinates the shape functions will be functions of ξ & η. In order to derive the stiffness matrices we need to evaluate the derivatives with respect to x and y. We therefore apply the chain rule to get

$$\frac{\partial N_i}{\partial \xi} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi} \quad \text{----- (1)}$$

$$\frac{\partial N_i}{\partial \eta} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \eta}$$

or in Matrix notation

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{pmatrix} \begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial Ni}{\partial \xi} \\ \dfrac{\partial Ni}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial Ni}{\partial x} \\ \dfrac{\partial Ni}{\partial y} \end{Bmatrix} \qquad \text{------ (2)}$$

Here 'J' is the Jocobian of transformation from Cartesian to Gaussian space. This gives the relationship between the derivatives of N_i with respect to the global and local co-ordinates.

From (2) we obtain

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial x} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix} \qquad \text{------ (3)}$$

Hence the Jacobian Martrix [J] must be non-singular

$$[J] = \begin{pmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{pmatrix} \qquad \text{------------------(4)}$$

We know that $x = \sum_{i=1} N_i(\xi,\eta)\, x_i$ ----- (5)

$$y = \sum_{i=1}^{m} N_i(\xi,\eta)\, y_i$$

$$\therefore \frac{\partial x}{\partial \xi} = \sum_{i=1}^{m} x_i \frac{\partial N_i}{\partial \xi} \qquad \frac{\partial y}{\partial \xi} = \sum_{i=1}^{m} y_i \frac{\partial N_i}{\partial \xi} \quad \text{---- (6)}$$

$$\frac{\partial x}{\partial \eta} = \sum_{i=1}^{m} x_i \frac{\partial N_i}{\partial \eta} \qquad \frac{\partial y}{\partial \eta} = \sum_{i=1}^{m} y_i \frac{\partial N_i}{\partial \eta}$$

Substituting equation (6) in (4) we get

$$[J]=\begin{pmatrix} \sum x_i \dfrac{\partial N_i}{\partial \xi} & \sum y_i \dfrac{\partial N_i}{\partial \xi} \\ \sum x_i \dfrac{\partial N_i}{\partial \eta} & \sum y_i \dfrac{\partial N_i}{\partial \eta} \end{pmatrix}$$

$$=\begin{pmatrix} \dfrac{\partial N_1}{\partial \xi} & \dfrac{\partial N_2}{\partial \xi} & \dfrac{\partial N_3}{\partial \xi} & \ldots\ldots & \dfrac{\partial N_m}{\partial \xi} \\ \dfrac{\partial N_1}{\partial \eta} & \dfrac{\partial N_2}{\partial \eta} & \dfrac{\partial N_3}{\partial \eta} & \ldots\ldots & \dfrac{\partial N_m}{\partial \eta} \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_m & y_m \end{pmatrix}$$

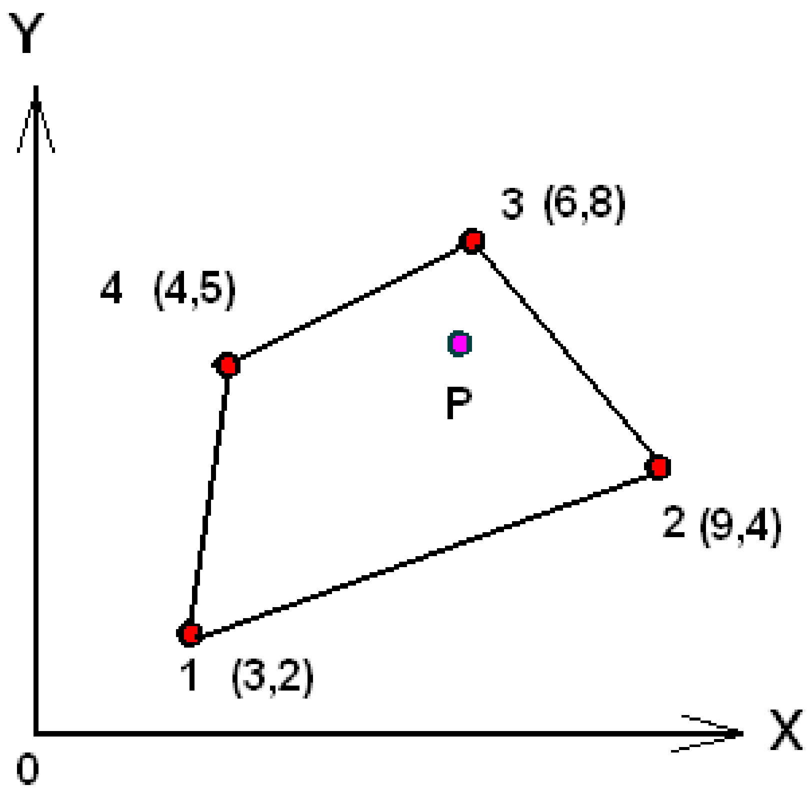
In general the Jacobian of transformation in 3D is given by

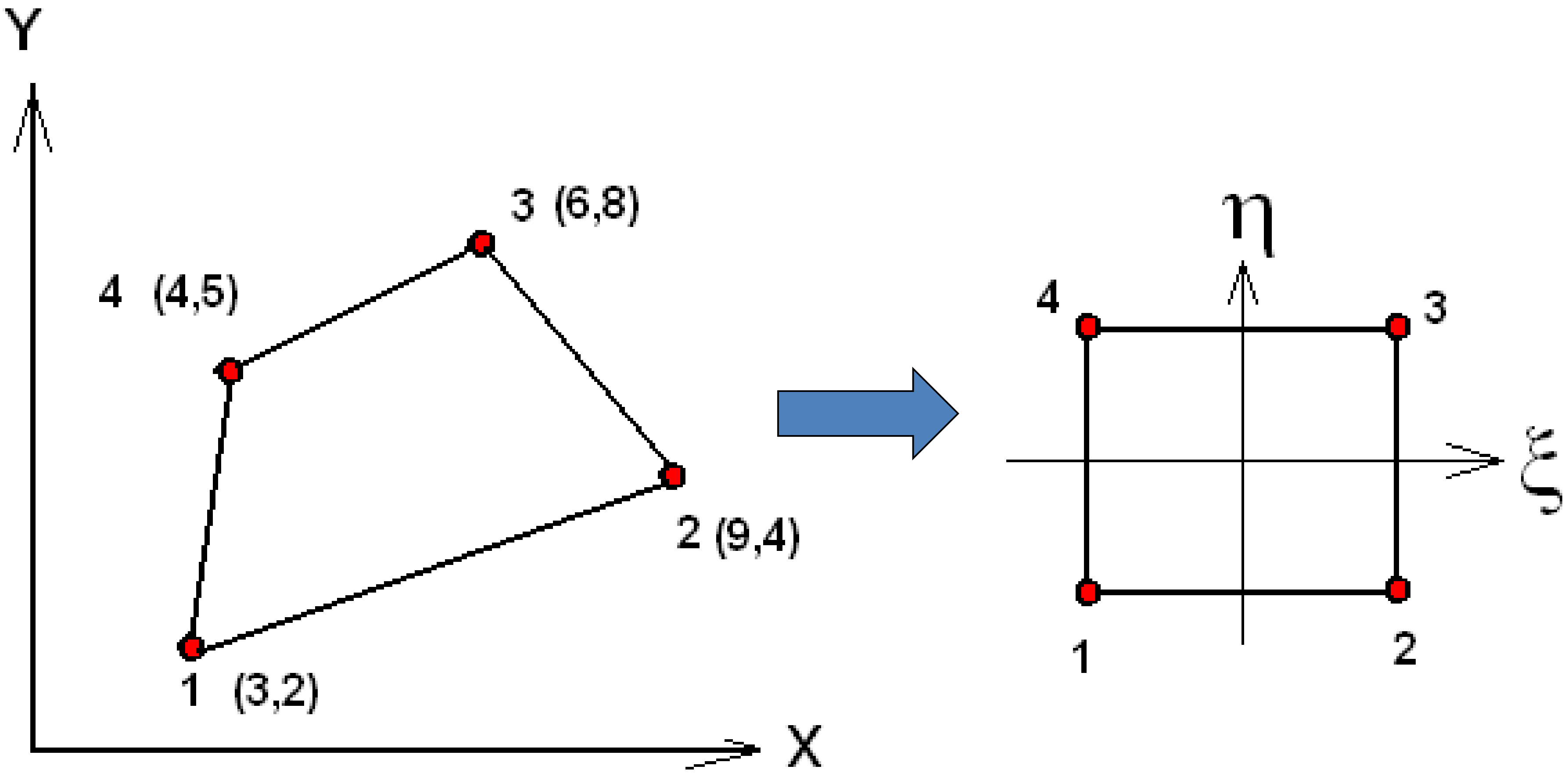
$$[J]=\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \\ \dfrac{\partial x}{\partial \psi} & \dfrac{\partial y}{\partial \psi} & \dfrac{\partial z}{\partial \psi} \end{bmatrix}$$

$$[J]=\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \\ \dfrac{\partial x}{\partial \psi} & \dfrac{\partial y}{\partial \psi} & \dfrac{\partial z}{\partial \psi} \end{bmatrix}$$

Problem:

Evaluate the Cartesian co-ordinate of the point P which has local co-ordinates $\xi = 0.6$ and $\eta = 0.8$ as shown in the Figure.




Y
X
4 (4,5)
3 (6,8)
2 (9,4)
1 (3,2)
η
ξ
4
3
1
2

Given: Natural co-ordinates of point P

$\xi = 0.6$

$\eta = 0.8$

Cartesian co-ordinates of point 1,2,3 and 4

$$x_1 = 3; \qquad y_1 = 2$$

$$x_2 = 9; \qquad y_2 = 4$$

$$x_3 = 6; \qquad y_3 = 8$$

$$x_4 = 4; \qquad y_4 = 5$$

To Find: The Cartesian co-ordinates of the point P (x,y)

Solution:

Shape functions for quadrilateral element are,

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$$

Substituting the values

$$\Rightarrow N_1(0.6,0.8) = \frac{1}{4}(1-0.6)(1-0.8) = 0.02$$

$$\Rightarrow N_2(0.6,0.8) = \frac{1}{4}(1+0.6)(1-0.8) = 0.08$$

$$\Rightarrow N_3(0.6,0.8) = \frac{1}{4}(1+0.6)(1+0.8) = 0.72$$

$$\Rightarrow N_4(0.6,0.8) = \frac{1}{4}(1-0.6)(1+0.8) = 0.18$$

$$Co-ordinate, x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$

$$= 0.02(3) + 0.08(9) + 0.72(6) + 0.18(4)$$

$$x = 5.82$$

$$Co-ordinate, y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$$

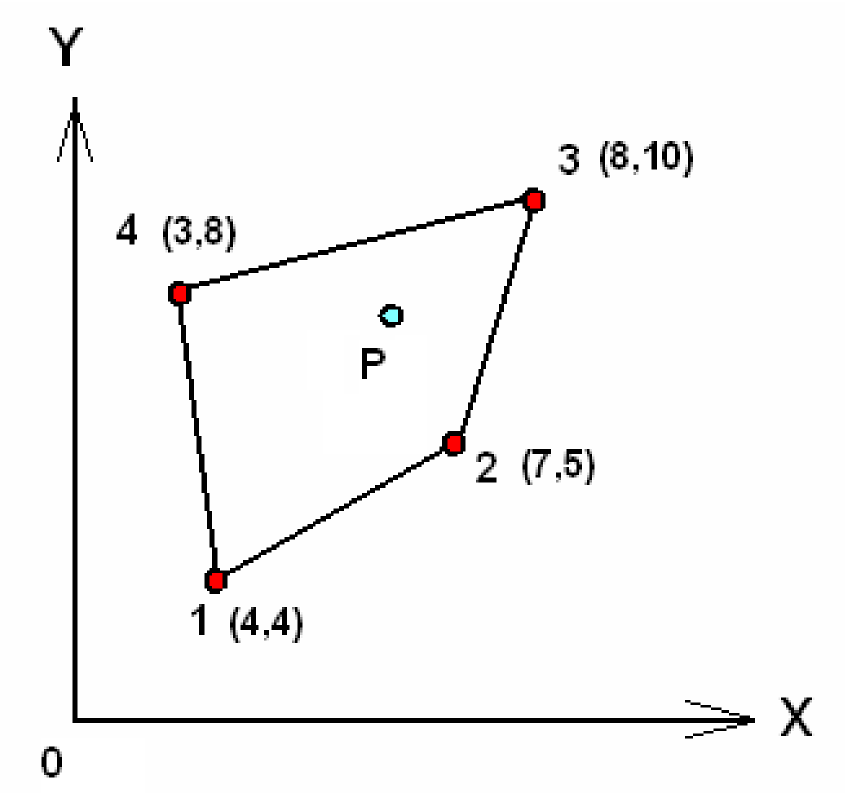
$$= 0.02 \times (2) + 0.08(4) + 0.72(8) + 0.18(5)$$

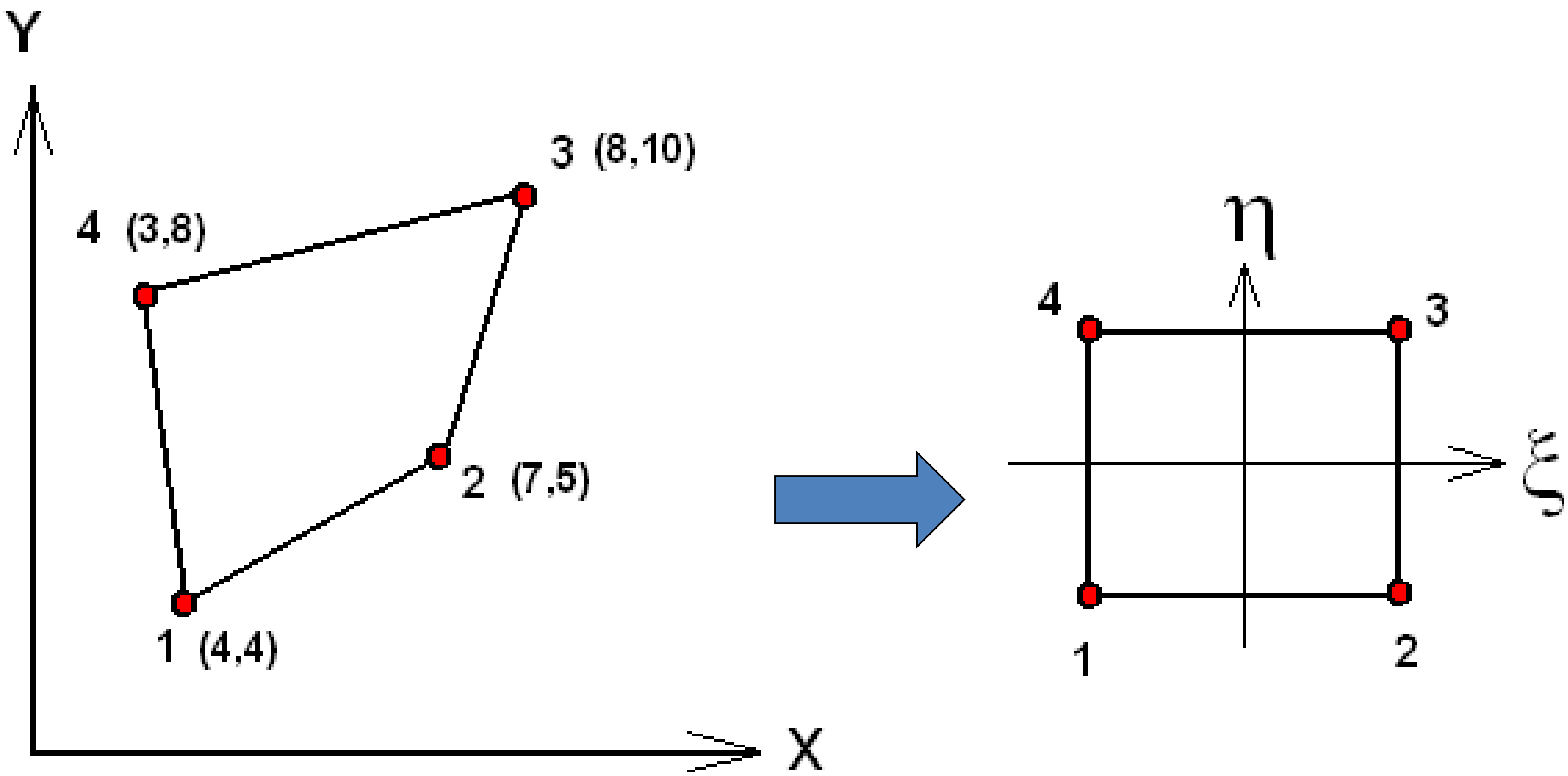
$$y = 7.02$$

$$Co-ordinates \ are \left((x, y) = (5.82, 7.02)\right)$$

Problem

Evaluate $[J]\, at\, \varepsilon = \eta = \frac{1}{2}$ **for the linear quadrilateral element shown in Fig.**





Y
3 (8,10)
4 (3,8)
2 (7,5)
1 (4,4)
X
η
4
3
ξ
1
2

Given:

Natural co-ordinates at point, P

$$\xi = \frac{1}{2} = 0.5\ ;\ \eta = \frac{1}{2} = 0.5$$

Cartesian co-ordinates of point 1,2,3 & 4

$$x_1 = 4; \qquad y_1 = 4$$

$$x_2 = 7; \qquad y_2 = 5$$

$$x_3 = 8; \qquad y_3 = 10$$

$$x_4 = 3; \qquad y_4 = 8$$

To Find: **1.Jacobian matrix [J].**

Solution: Jacobian matrix for quadrilateral element is given by,

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \qquad [J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{1}{4}[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4]$$

$$J_{12} = \frac{1}{4}[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4]$$

$$J_{21} = \frac{1}{4}[-(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4]$$

$$J_{22} = \frac{1}{4}[-(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4]$$

$$J_{11}(0.5,0.5)=\frac{1}{4}[-(1-0.5)4+(1-0.5)7+(1+0.5)8-(1+0.5)3]$$

$$=2.25$$

$$J_{12}(0.5,0.5)=\frac{1}{4}[-(1-0.5)4+(1-0.5)5+(1+0.5)10-(1+0.5)8]$$

$$=0.875$$

$$J_{21}(0.5,0.5)=\frac{1}{4}[-(1-0.5)4-(1+0.5)7+(1+0.5)8+(1-0.5)3]$$

$$=0.25$$

$$J_{22}(0.5,0.5)=\frac{1}{4}[-(1-0.5)4-(1+0.5)5+(1+0.5)10+(1-0.5)8]$$

$$=2.375$$

$$\Rightarrow [J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 2.25 & 0.875 \\ 0.25 & 2.375 \end{bmatrix}$$

Stiffness Matrix for a 2 Noded Axial Element

$$[K] = \int_0 B^T \, D \, BA dx$$

$$[B] = \frac{du}{dx} = \frac{dN}{dx} = \frac{1}{J}\frac{dN}{d\xi}$$

$$= \frac{2}{L}\begin{pmatrix} \frac{dN_1}{d\xi} & \frac{dN_2}{d\xi} \end{pmatrix}$$

$$= \frac{2}{L}\begin{pmatrix} \frac{d}{d\xi}\frac{(1-\xi)}{2} & \frac{d}{d\xi}\frac{(1+\xi)}{2} \end{pmatrix}$$

$$= \frac{2}{L}\begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{-1}{L} & \frac{1}{L} \end{pmatrix}$$

$$[K] = A \int_{-1}^{+1} \begin{Bmatrix} -1/L \\ 1/L \end{Bmatrix} E \langle -1/L \;\; 1/L \rangle \; J \, d\xi$$

$$= EA \int_{-1}^{+1} \begin{pmatrix} -1/L \\ 1/L \end{pmatrix} \langle -1/L \;\; 1/L \rangle \; L/2 \, d\xi$$

$$= \frac{EAL}{2} \int_{-1}^{+1} \begin{pmatrix} 1/L^2 & -1/L^2 \\ -1/L^2 & 1/L^2 \end{pmatrix} d\xi$$

$$= \frac{EA}{2L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \int_{-1}^{+1} d\xi = \frac{EA\,\cancel{2}}{\cancel{2}L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Problem:

For the four noded rectangular element shown if Fig. determine the following:

i) Jacobian matrix

ii) Strain-Displacement matrix

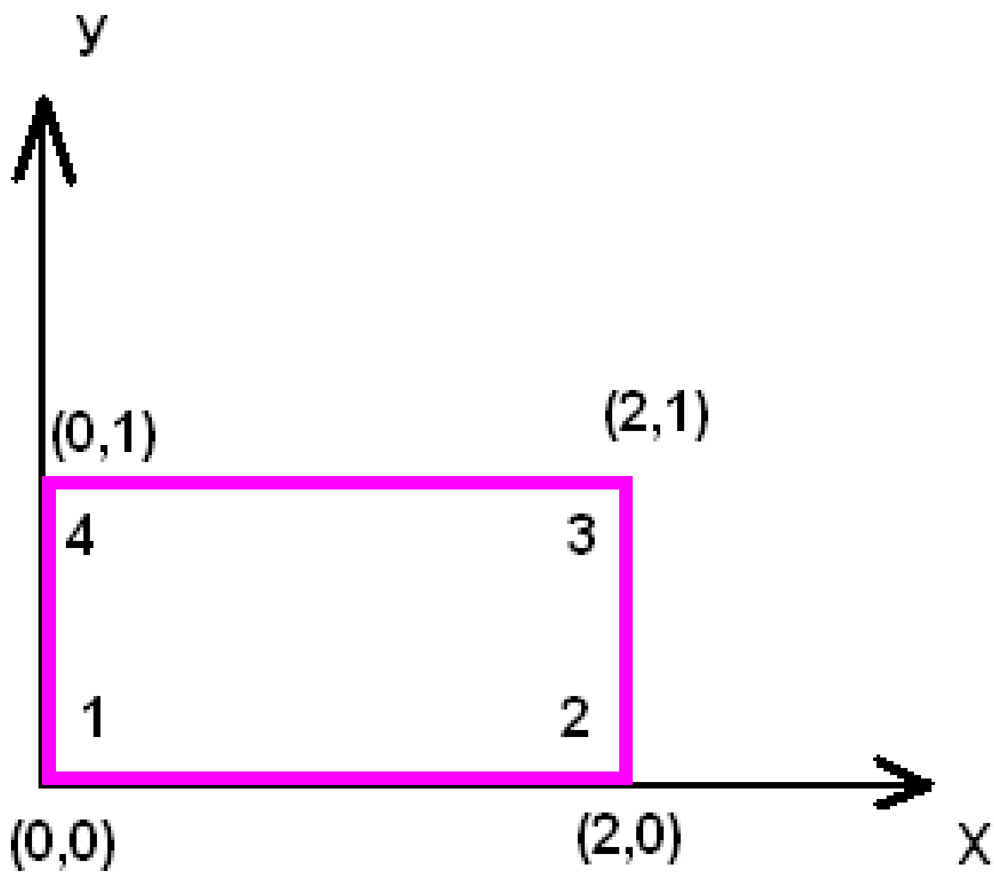
iii)Element stresses

Take $E = 2 \times 10^5$ N/mm^2; $v = 0.25$;

$u = [0, 0, 0.003, 0.004, 0.006, 0.004, 0, 0]^T$

$\xi = 0$; $\eta = 0$

Assume plane stress condition.


y
(0,1)
(2,1)
4
3
1
2
(0,0)
(2,0)
X

Cartesian co-ordinates of point 1,2,3 & 4

$$x_1 = 0; \qquad y_1 = 0$$

$$x_2 = 2; \qquad y_2 = 0$$

$$x_3 = 2; \qquad y_3 = 1$$

$$x_4 = 0; \qquad y_4 = 1$$

Young’s modulus, E = 2 x 10^5 N/mm^2

Poisson’s ratio *v* = 0.25

$$Displacement, u = \begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

$$Natural\ Co-ordinates,\ \xi = 0,\ \eta = 0$$

To Find: 1. Jacobian matrix, J.

2. Strain Displacement, [B]

3. Element stress, σ

Solution:

$$[J]=\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11}=\frac{1}{4}[-(1-\eta)x_1+(1-\eta)x_2+(1+\eta)x_3-(1+\eta)x_4]$$

$$J_{12}=\frac{1}{4}[-(1-\eta)y_1+(1-\eta)y_2+(1+\eta)y_3-(1+\eta)y_4]$$

$$J_{21}=\frac{1}{4}[-(1-\varepsilon)x_1-(1+\varepsilon)x_2+(1+\varepsilon)x_3+(1-\varepsilon)x_4]$$

$$J_{22}=\frac{1}{4}[-(1-\varepsilon)y_1-(1+\varepsilon)y_2+(1+\varepsilon)y_3+(1-\varepsilon)y_4]$$

$$J_{11}(0,0) = \frac{1}{4}[0+2+2-0] \quad ; \quad J_{12}(0,0) = \frac{1}{4}[0+0+1-1]$$

$$= 1 \qquad\qquad\qquad\qquad = 0$$

$$J_{21}(0,0) = \frac{1}{4}[0-2+2+0] \quad ; \quad J_{22}(0,0) = \frac{1}{4}[-0-0+1+]$$

$$= 0 \qquad\qquad\qquad\qquad = 0.5$$

$$\Rightarrow [J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$Jacobian \;\; matrix, [J] = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\Rightarrow |J| = 1 \times 0.5 - 0 \;\; = 0.5$$

Strain- Displacement matrix for quadrilateral element is,

$$\Rightarrow [B] = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times \frac{1}{4}$$

$$\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) \end{bmatrix}$$

$$\Rightarrow [B] = \frac{1}{0.5}\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0.5 & 0 \end{bmatrix} \times \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{0.5\times 4}\begin{bmatrix} -0.5 & 0 & 0.5 & 0 & 0.5 & 0 & -0.5 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & -0.5 & -1 & 0.5 & 1 & 0.5 & 1 & -0.5 \end{bmatrix}$$

$$= \frac{0.5}{0.5\times 4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

$$[B] = 0.25 \begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

Element stress, σ = [D] [B] {u}

$$Stress - strain\ relationship\, matrix,$$

$$[D] = \frac{E}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix}$$

$$= \frac{2\times10^5}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \dfrac{1-0.25}{2} \end{bmatrix}$$

$$= 213.33\times10^3\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$= 213.33\times10^3\times0.25\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D] = 53.33 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

Substituting the values in Element stress equation

$$\sigma = [D][B]\{d\}$$

$$\Rightarrow \{\sigma\} = 53.33\times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$\times 0.25 \begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

$$\times \begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

$$=53.33\times10^{3}\times0.25\begin{bmatrix}-4 & -2 & 4 & -2 & 4 & 2 & -4 & 2\\ -1 & -8 & 1 & -8 & 1 & 8 & -1 & 8\\ -3 & -1.5 & -3 & 1.5 & 3 & 1.5 & 3 & -1.5\end{bmatrix}$$

$$\times\begin{Bmatrix}0\\0\\0.003\\0.004\\0.006\\0.004\\0\\0\end{Bmatrix}$$

$$\{\sigma\} = 13.333 \times 10^3 \begin{Bmatrix} 0.036 \\ 0.009 \\ 0.021 \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} 480 \\ 120 \\ 280 \end{Bmatrix} N/m^2$$

NUMERICAL INTEGRATION

In the isoparametric formulation of higher order elements we see that the strain-displacement matrix [B] is given by

$$[B] = \frac{du}{dx} = \frac{dN}{dx}[\xi] = \frac{1}{J}\frac{d[N]}{d\xi}$$

$$= \frac{1}{J}\left(\frac{d}{d\xi}\left(\frac{-\xi+\xi^2}{2} \quad 1-\xi^2 \quad \frac{\xi+\xi^2}{2}\right)\right)$$

Here $J = \left(\frac{-1 + 2\xi}{2} \quad -2\xi \quad \frac{1 + 2\xi}{2} \right)$

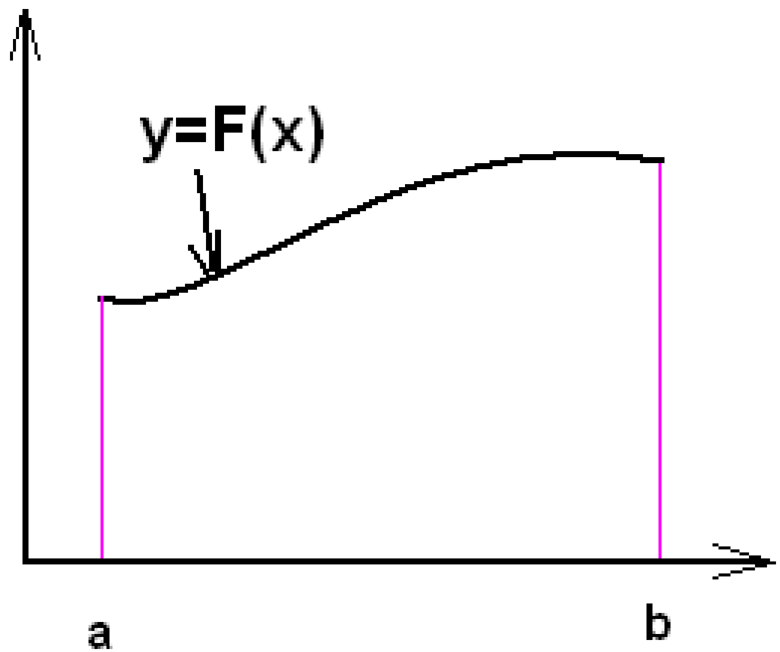
Therefore Matrix [B] is a function of ξ, with polynomials in ξ in its denominator because of the 1/J factor. Hence the equation (A) cannot be integrated to give on the solution. Hence we resort to numerical integration.

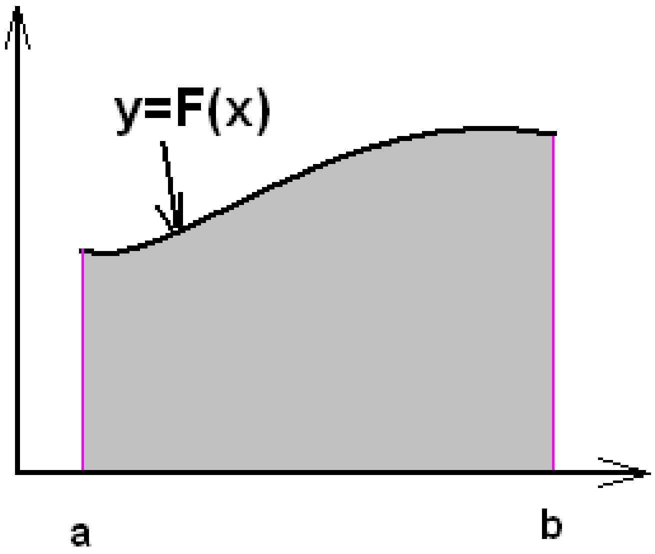
So evaluation of integrals of the form

$$\int_a^b F(x)\,dx$$ becomes difficult or impossible in cases where the integrand F has functions of x in both numerator denominator.

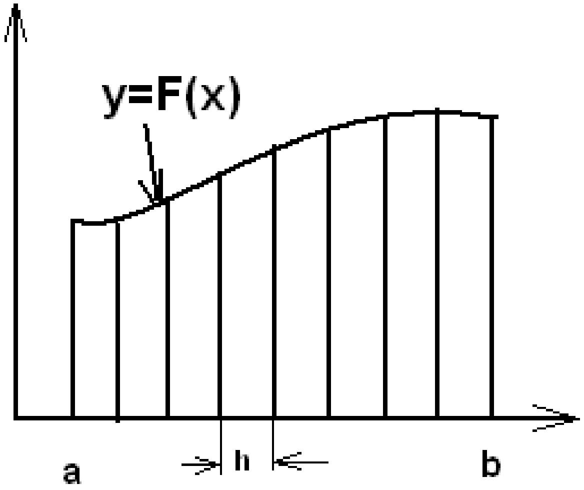
The basic idea behind whatever numerical integration technique we may employ is that of obtaining a function P(x) which is both a suitable approximation of F(x) and simple enough to integrate.

Referring to Fig the variation of F(x) is shown. Evaluation of the Integral $\int_a^b F(x)\,dx$ will yield the area under the F(x) curve between points x_1 (= a) & x_2 (= b).

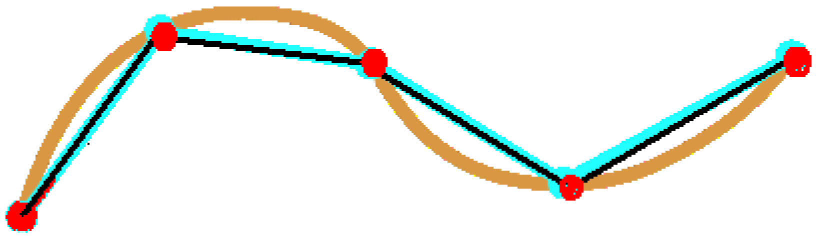


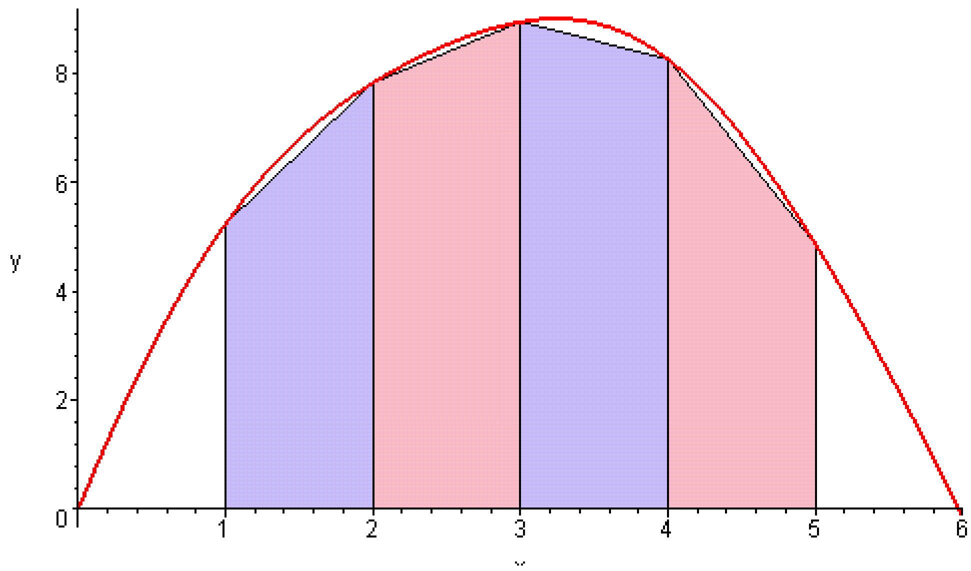


“<u>Trapezoidal rule</u>”,

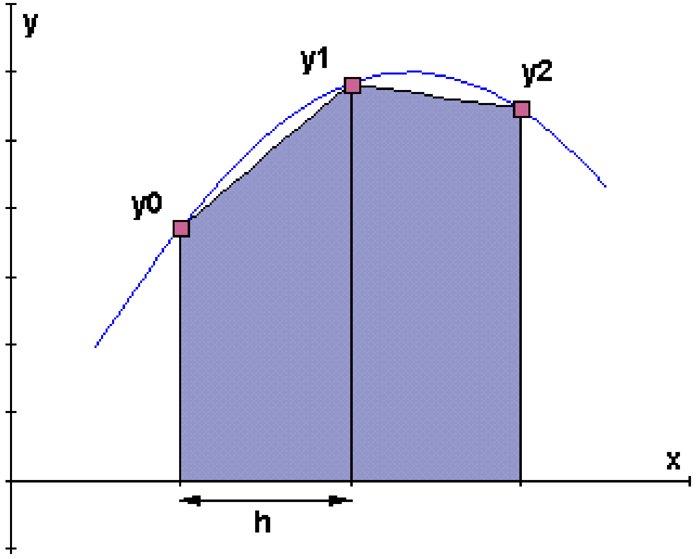


$$\int_a^b F(x)dx = \frac{h}{2}(y_0 + y_8 + (y_1 + y_2 + \ldots\ldots.y_7))$$

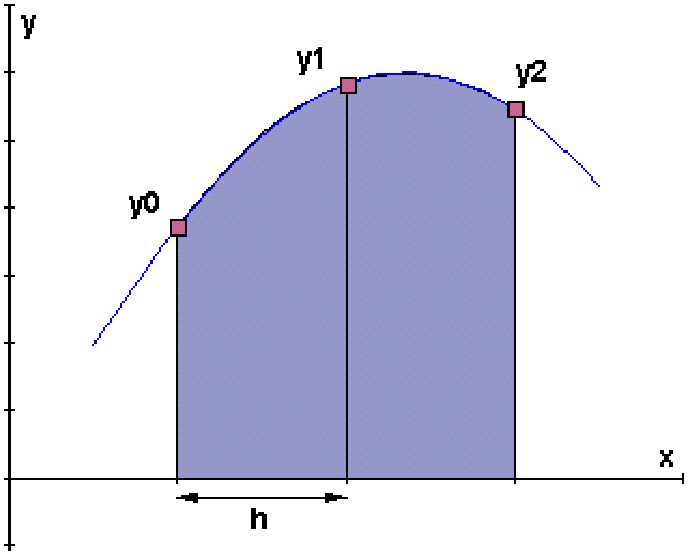




y
0
2
4
6
8
1
2
3
4
5
6
x

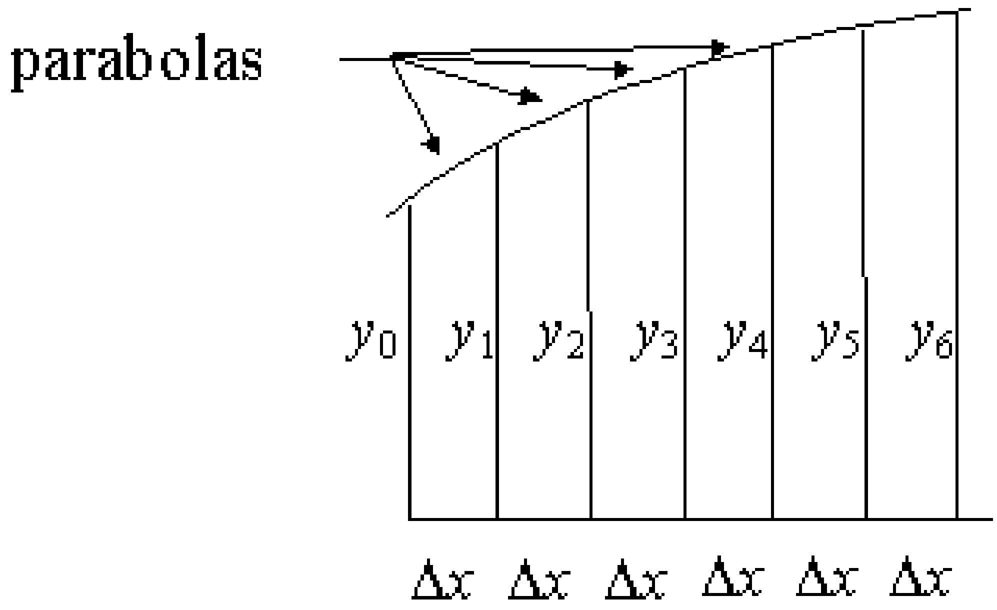


Trapezoidal Rule



Simpsons Rule

$$\int_a^b F(x)dx = \frac{h}{3}(y_0 + y_8 + 4(y_1 + y_3 + \dots\dots.. y_7) + 2(y_2 + y_4 + \dots\dots.. y_6)$$



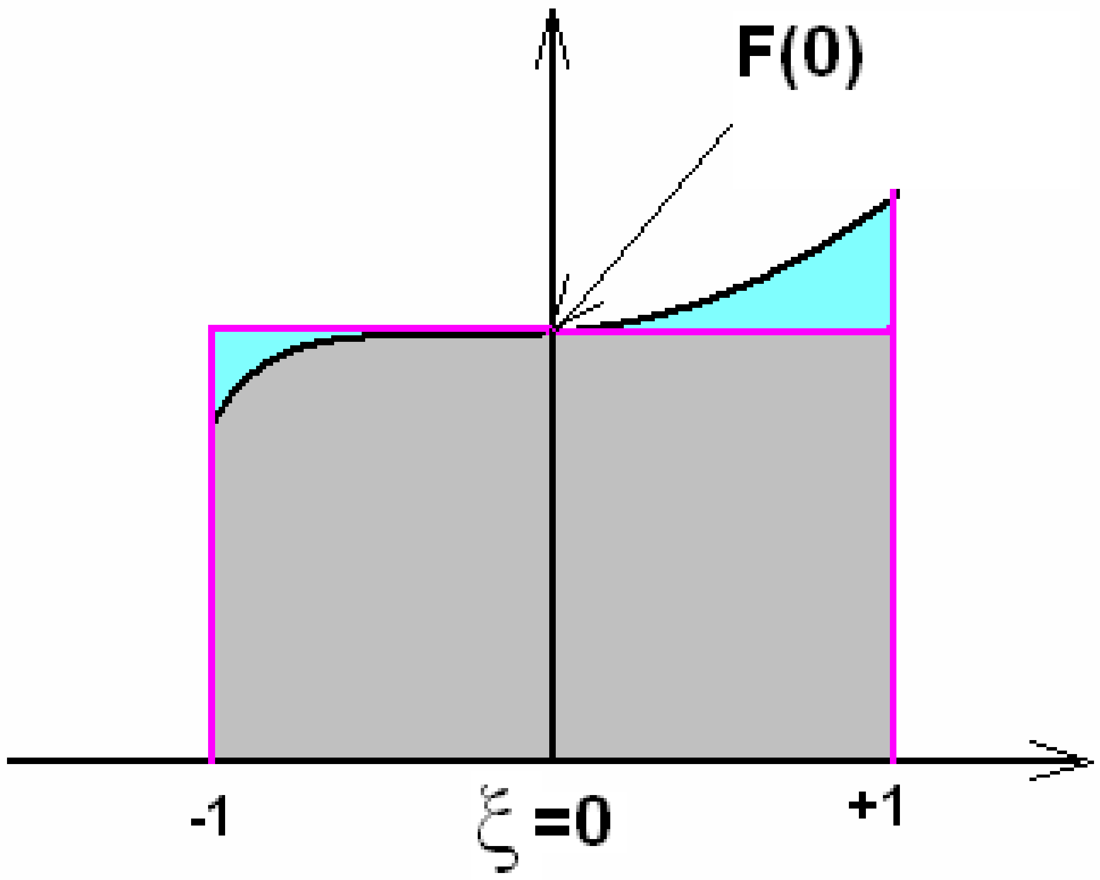
Gauss Quadrature:- Amongst the several schemes available for evaluating the area under the curve F(x) between two points the gauss quadrature method has proved to be most useful for isoparametric elements. As in isoparametric formulation, the limits of the integral are always from –1 to +1, the problem in gauss integration is to evaluate the integral

$$I = \int_{-1}^{+1} F(\xi)\, d\xi.$$

The simplest and probably the crudest way to evaluate the integral is to sample or evaluate $F(\xi)$ at the mid point of the interval and to multiply this by the length of the element which is '2' [because $\xi_1 = -1$ & $\xi_2 = 1$ & $(\xi_2 - \xi_1) = 2$]

$$\therefore \int F(x)\, dx = I = 2\, f_i$$

This result will be exact only if the actual function happens to be a straight line.



One point formula

We can extend the same to take two sampling points or three etc.Generalization of this relation gives

$$I = \int_{-1}^{+1} F(\xi)\, d\xi = w_1 f_1 + w_2 f_2 + \ldots.. w_n f_n$$

$$= \sum_{i=1}^{n} w_i f(\xi_i)$$

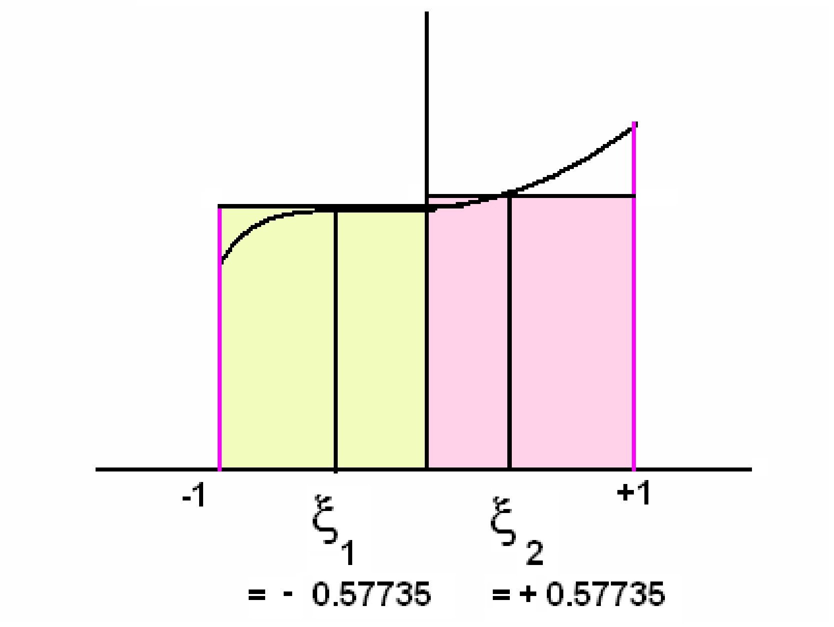
Here w_i is called the 'weight' associated with the i^{th} point and n is the number of sampling points. The Table (1) gives the sampling points and the associated weights (w_i) for Gauss quadrature.

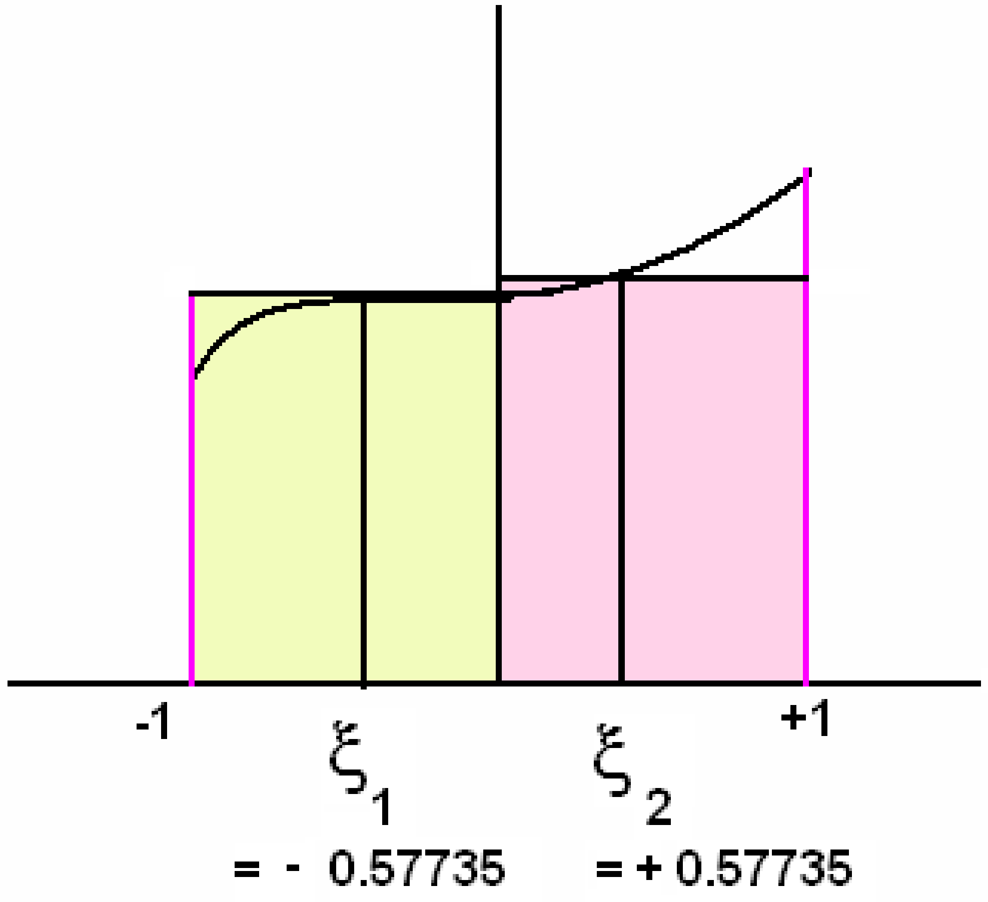
No.of points	Location	Weight W_i
1	$\xi_1 = 0.00000$	2.00000
2	$\xi_1, \xi_2 = \pm 0.57735$	1.000000
3	$\xi_1, \xi_3 = \pm 0.77459$ $\xi_2 = 0.00000$	0.55555 0.88888
4	$\xi_1, \xi_4 = \pm 0.8611363$ $\xi_2 \xi_3 = \pm 0.3399810$	0.3478548 0.6521451

Thus to approximate the integral I, the function $f(\xi)$ is evaluated at each of several locations ξ_i, and each $f(\xi_i)$ is multiplied by the approximating weights w. The summation of these products gives the value of the integral. The sampling points are generally located symmetrically with respect to the center of the interval. Symmetrically paired points have the same weight w_i.

As an example consider the evaluation of the Integral $\boldsymbol{I}$ using 2 sampling points i.e. n = 2.

$I \approx$ (1.0) (f at ξ = - 0.577350269189626) + (1.0) (f at ξ = + 0.577350269189626)

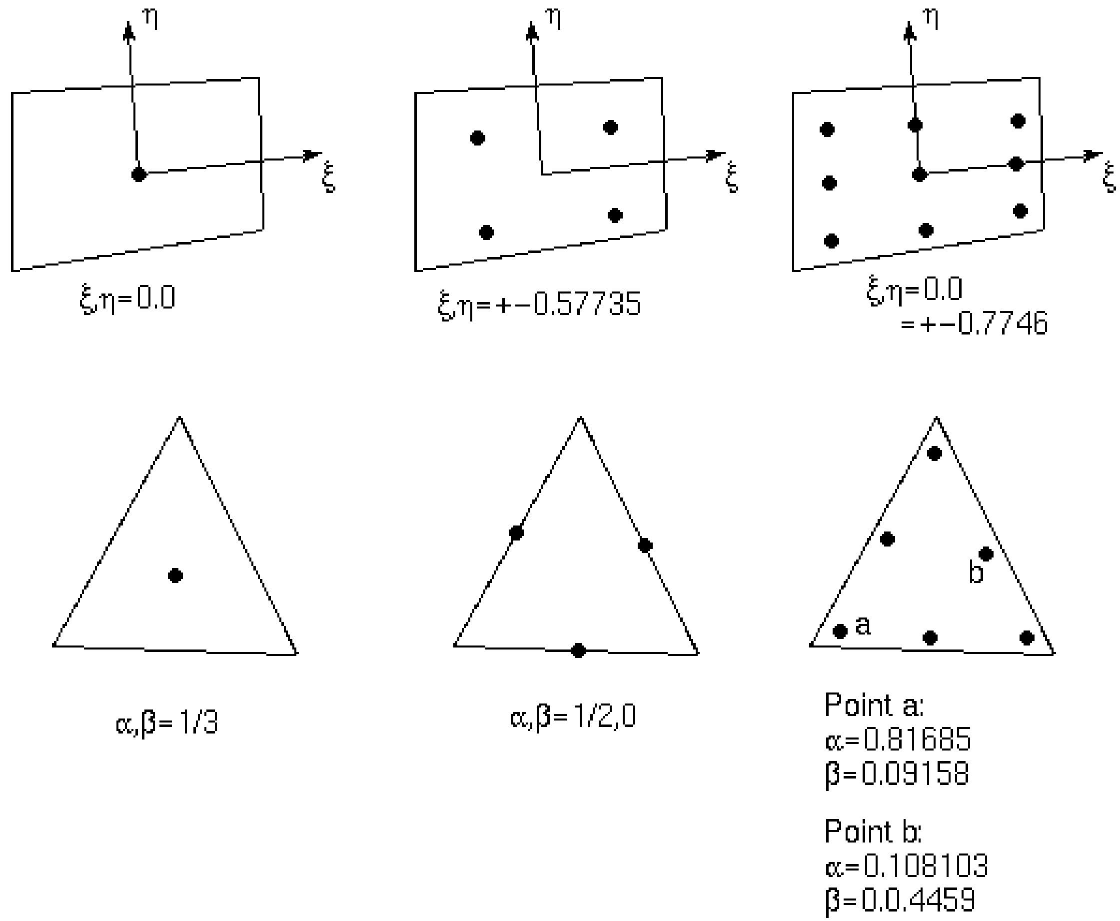



-1
+1
ξ_1
$= -\ 0.57735$
ξ_2
$= +\ 0.57735$

In general if we know that the integral to be evaluated is of order p then the number of sampling points required n is given by the relation

$$2n-1 = p$$

The calculated number of sampling points can be rounded off to the nearest integer


η
ξ
ξ,η=0.0
ξ,η=+−0.57735
ξ,η=0.0
=+−0.7746
α,β=1/3
α,β=1/2,0
a
b
Point a:
α=0.81685
β=0.09158
Point b:
α=0.108103
β=0.0.4459

Problem

Evaluate the integral $I = \int_{-1}^{1} (2 + x + x^2)\,dx$ **and compare with exact solution.**

Given: Integral, $I = \int_{-1}^{1} (2 + x + x^2)\,dx$

$$\Rightarrow f(x) = 2 + x + x^2$$

To Find: The integral I by using Gauss quadrature.

Solution:

We know that , the given integrand is a polynomial of order 2.

So, 2n-1 = 2

$\Rightarrow$ **2n =3**

$\Rightarrow$ **n = 1.5 ≈ 2**

For two point Gaussian quadrature,

$$x_1 = +\sqrt{\frac{1}{3}} = 0.577350269 \qquad w_1 = 1$$

$$x_2 = -\sqrt{\frac{1}{3}} = -0.577350269 \qquad w_2 = 1$$

$$f(x) = 2 + x + x^2$$

$$f(x_1) = 2 + x_1 + x_1^2$$

$$= 2 + (0.577350269) + (0.577350269)^2$$

$$f(x_1) = 2.9106836$$

$$w_1 \ f(x_1) = 1 \times 2.9106836$$

$$\Rightarrow w_1 \ f(x_1) = 2.9106836$$

$$f(x_2) = 2 + x_2 + x_2^2$$

$$= 2 - (0.577350269) + (-0.577350269)^2$$

$$f(x_2) = 1.755983$$

$$w_2 f(x_2) = 1 \times 1.755983$$

$$w_2 f(x_2) = 1.755983$$

$$w_1 f(x_1) + w_2 f(x_2) = 2.9106836 + 1.755983$$

$$= 4.666666$$

$$\Rightarrow \int_{-1}^{1} (2 + x + x^2)\, dx = 4.666666$$

Exact Solution:

$$\int_{-1}^{1}(2+x+x^2)\,dx = 2\big[x\big]_{-1}^{+1} + \frac{1}{2}\big[x^2\big]_{-1}^{+1} + \frac{1}{3}\big[x^3\big]_{-1}^{+1}$$

$$= 2[1-(-1)] + \frac{1}{2}[1-(1)] + \frac{1}{3}[1-(-1)]$$

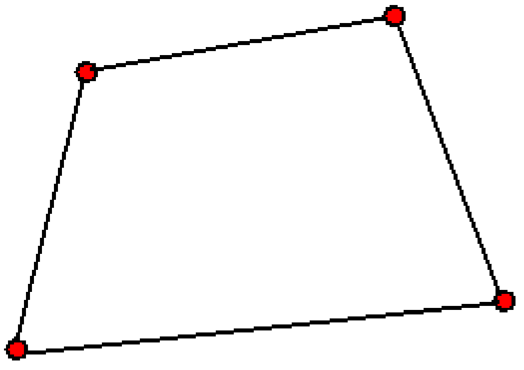
$$= 4.666666$$

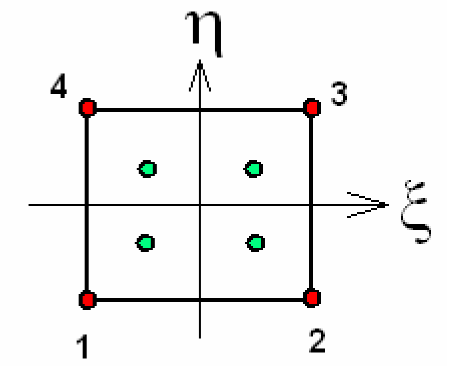
Using Gauss Quadrature evaluate the following integral using 1 2 and 3 point Integration.

i) $$\int_{-1}^{1} \frac{\sin s}{s\,(1-s^2)}\,ds$$

ii) $$\int_{-1}^{1} \frac{\cos^2 s}{s\,(1-s^2)}\,ds$$

iii) $$\int_{-1}^{1} \frac{r^2-1}{(r+3)^2}\,dr$$





ξ,η=0.57735

$$F(\xi,\eta) = \sum_{i=1}^{n} f(\xi_i, \eta_i)\, w_i w_j$$